Finite and Algorithmic Model Theory

Lecture 1 (Dresden 12.10.22)

Lecturer: Bartosz “Bart” Bednarczyk

Technische Universität Dresden & Uniwersytet Wrocławski

TECHNISCHE UNIVERSITÄT DRESDEN

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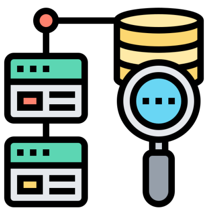

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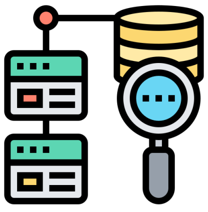

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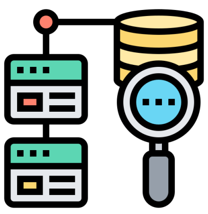

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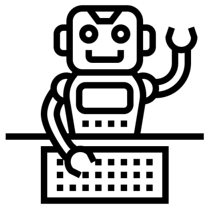

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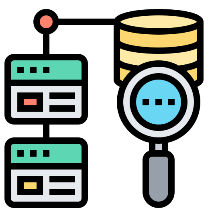

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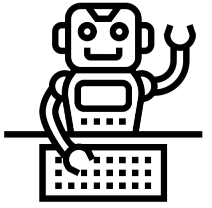

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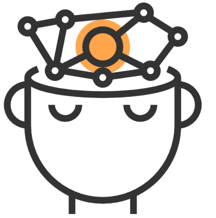

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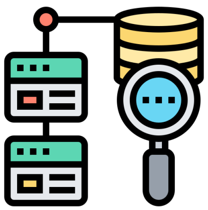

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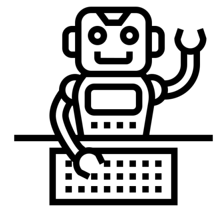

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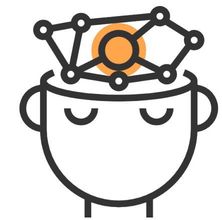

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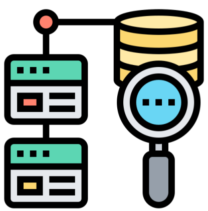

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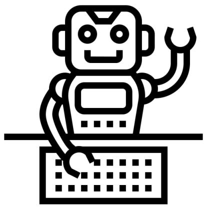

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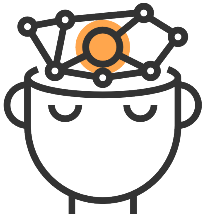

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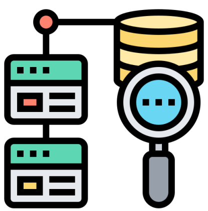

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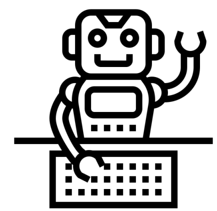

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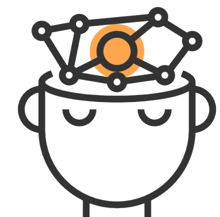

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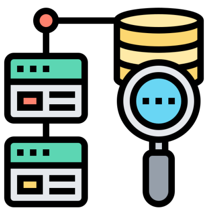

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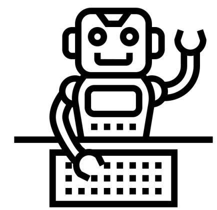

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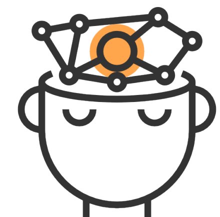

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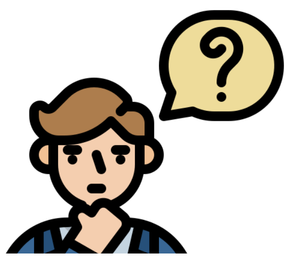

Feel free to ask questions and interrupt me!

Don't be shy! If needed send me an email (bartosz.bednarczyk@cs.uni.wroc.pl) or approach me after the lecture!

Reminder: this is an advanced lecture. Target: people that had fun learning logic during BSc studies!

Course Information

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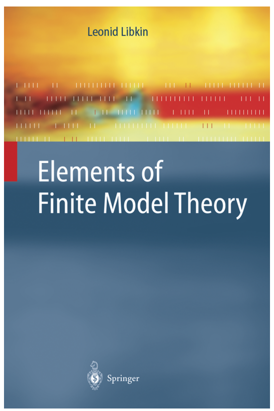

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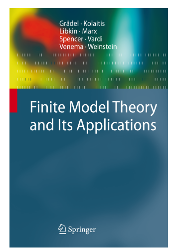

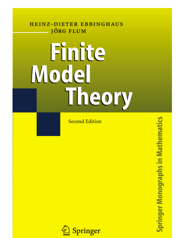

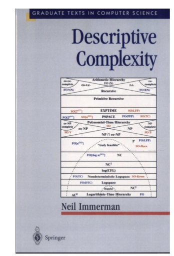

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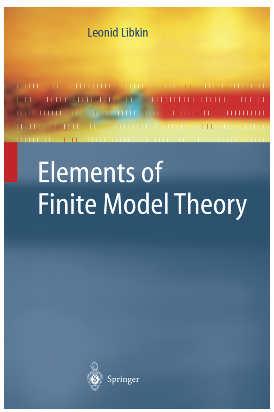

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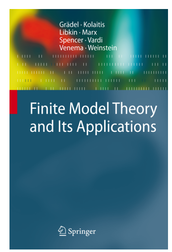

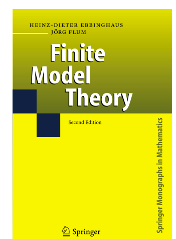

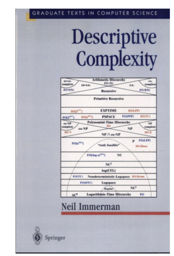

Last but Not Least: I offer MSc/PHD research projects for motivated students!

2–3. Examples and Motivations

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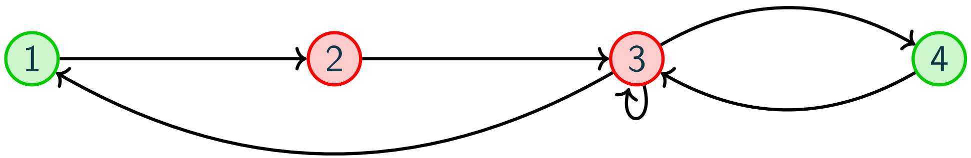
$\mathfrak{A} :=$ 

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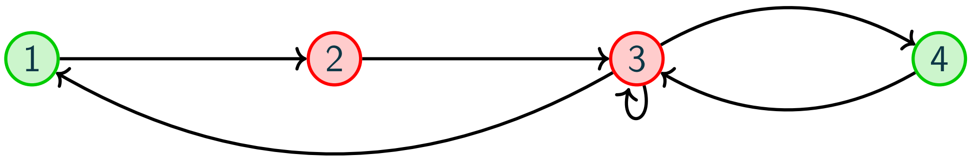
$\mathfrak{A} :=$ 

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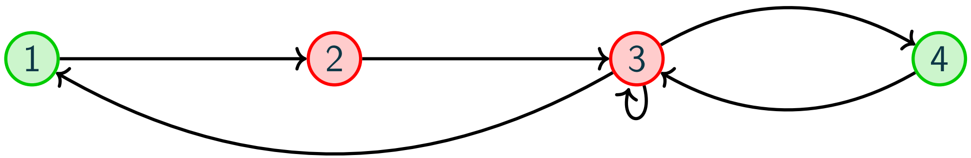
$\mathfrak{A} :=$ 

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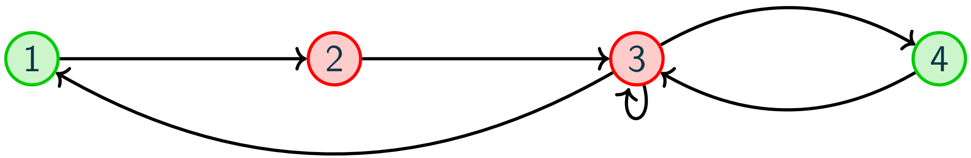

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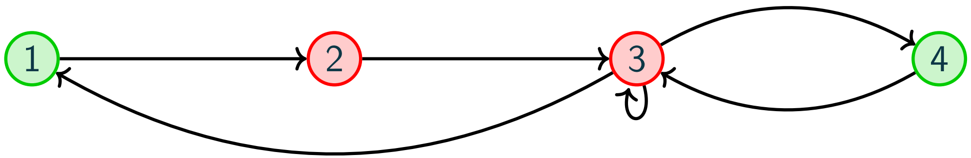
$\mathfrak{A} :=$ 

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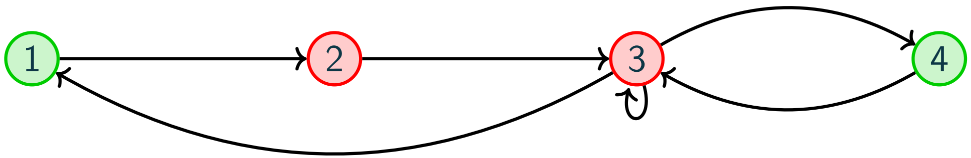
$\mathfrak{A} :=$ 

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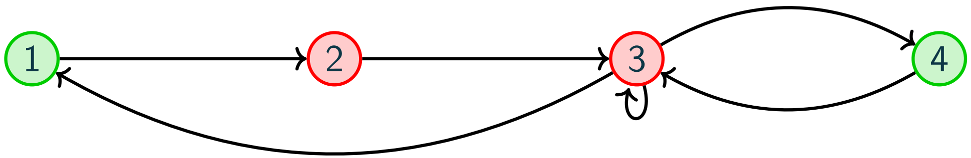
$\mathfrak{A} :=$ 

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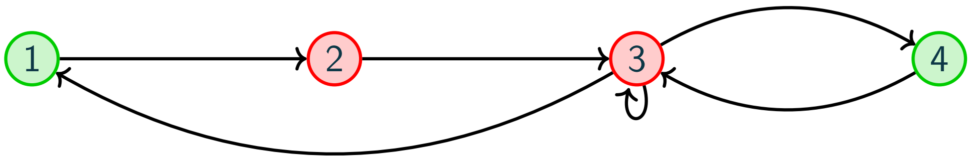
$\mathfrak{A} :=$ 

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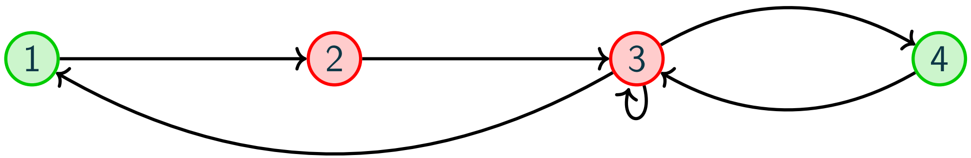
$\mathfrak{A} :=$ 

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Constants $\approx$ elements, unary relations $\approx$ colours, binary (resp. higher-arity) relations $\approx$ (hyper)edges

What is a “logic”? A running example.

Naively: a “formal language” for expressing properties of relational structures ($\approx$ hypergraphs).

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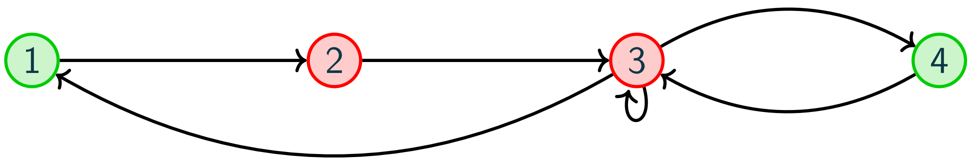
$\mathfrak{A} :=$ 

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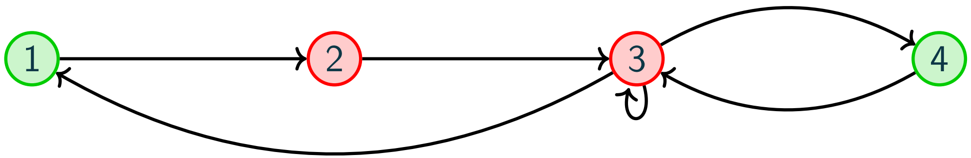
$\mathfrak{A} :=$ 

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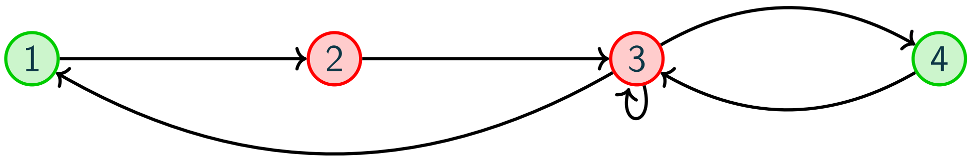

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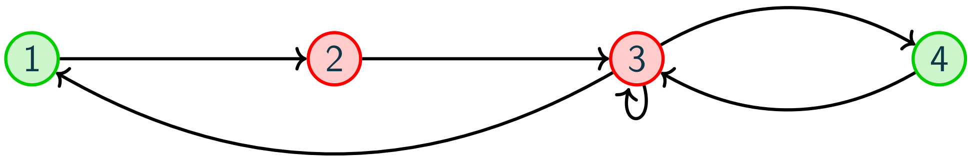

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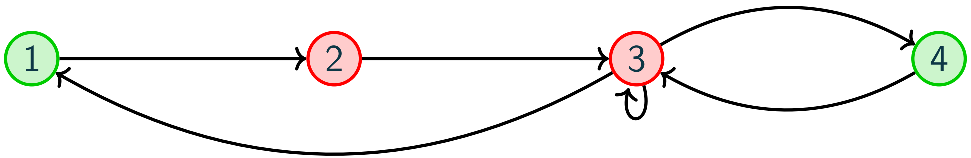
$\mathfrak{A} :=$ 

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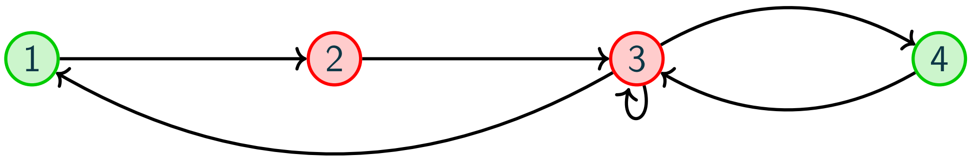
$\mathfrak{A} :=$ 

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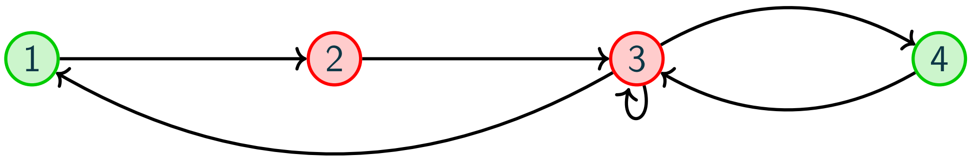
$\mathfrak{A} :=$ 

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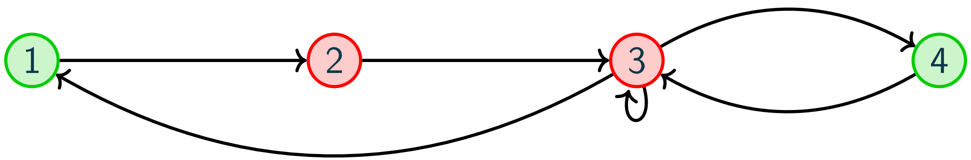

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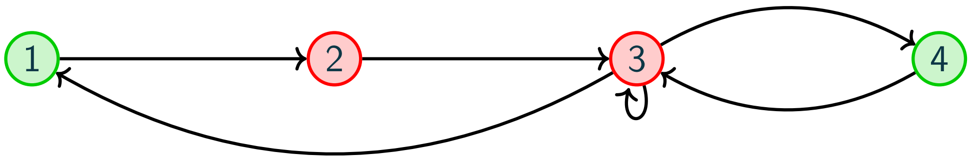
$\mathfrak{A} :=$ 

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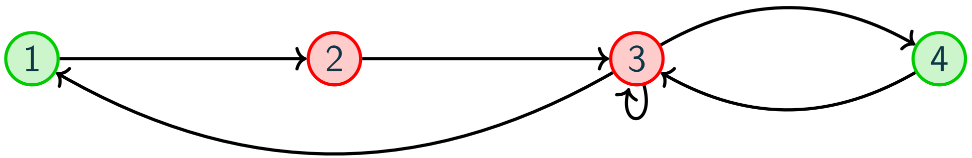
$\mathfrak{A} :=$ 

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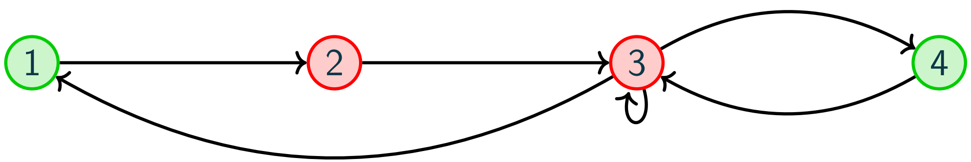

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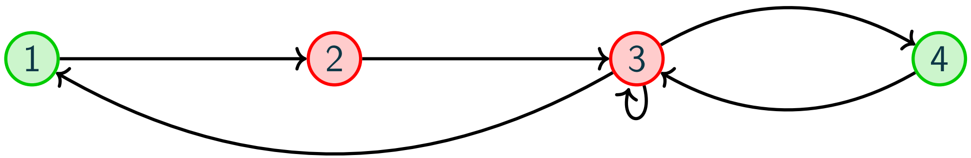

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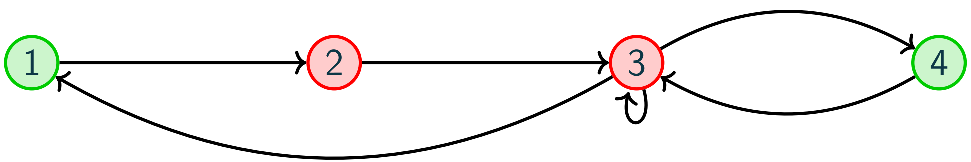
$\mathfrak{A} :=$ 

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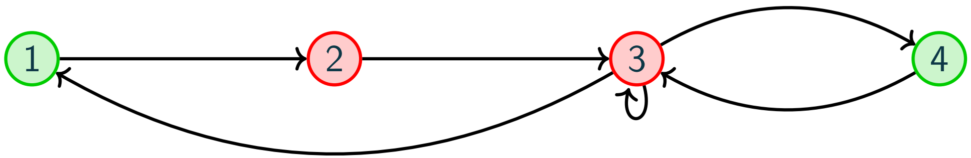
$\mathfrak{A} :=$ 

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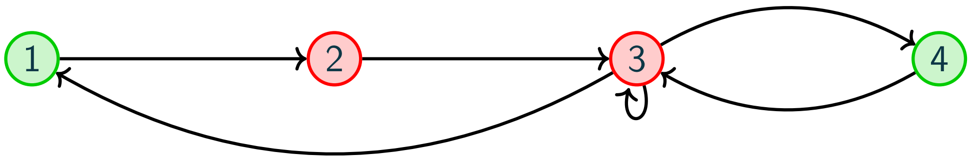
$\mathfrak{A} :=$ 

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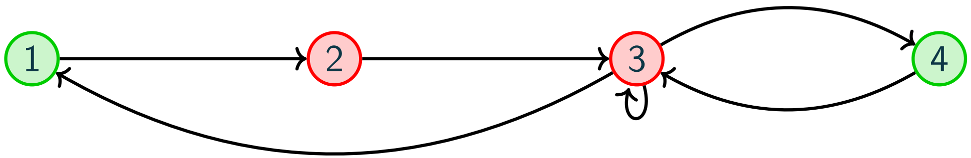
$\mathfrak{A} :=$ 

over a signature $\tau := \{G^{(1)}, R^{(1)}, E^{(2)}\}$

$G^{\mathfrak{A}} := \{1, 4\}, \quad R^{\mathfrak{A}} := \{2, 3\}$

$E^{\mathfrak{A}} := \{(1,2),(2,3),(3,1),(3,3)(3,4),(4,3)\}$

A signature contains (at most countably* many) constant and relation symbols (each with a fixed arity).

Structure = Domain + interpretation of symbols, e.g. $\mathfrak{A} := (A, \cdot^{\mathfrak{A}})$ depicted above,

where $A = \{1,2,3,4\}$ and $\cdot^{\mathfrak{A}}(G), \cdot^{\mathfrak{A}}(R), \cdot^{\mathfrak{A}}(E)$ are as above.

Example (of a First-Order Logic (FO) Formula)

(in a coloured graph:) Any node is either green or red.

$\varphi := \forall x\, (G(x) \vee R(x)) \wedge (G(x) \leftrightarrow \neg R(x))$

We write $\mathfrak{A} \models \varphi$ to indicate that $\mathfrak{A}$ satisfies φ or $\mathfrak{A}$ is a model of φ.

Formulae often employ: Variables: $x, y, z, X, Y, \ldots$ Boolean connectives: $\wedge, \vee, \neg, \leftrightarrow, \bigvee_{i=0}^{\infty}, \ldots$

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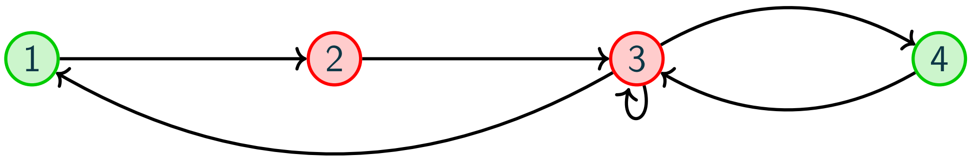
$\mathfrak{A} :=$ 

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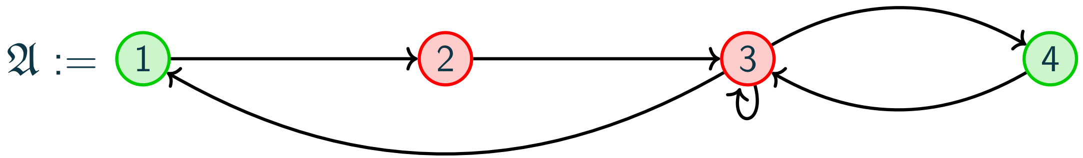

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More examples I.

More examples I.

Exercise (An FO[$\{\mathrm{E}^{(2)}\}$] formula/query testing if a graph is a 4-element clique [here E = edge relation].)

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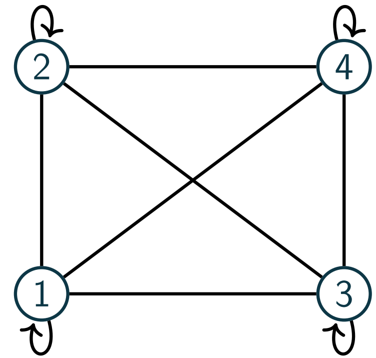

$\models \varphi_{\mathsf{4clique}}$

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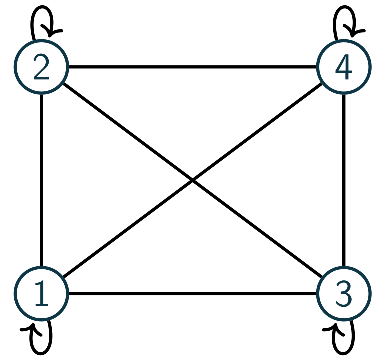

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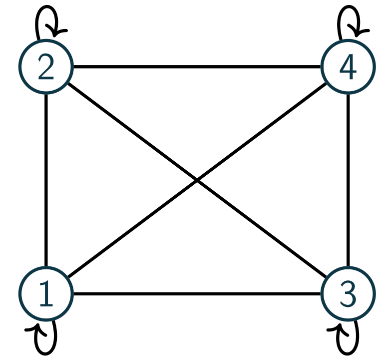

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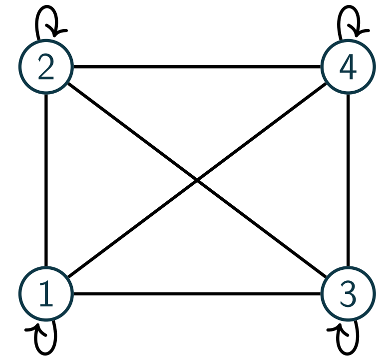

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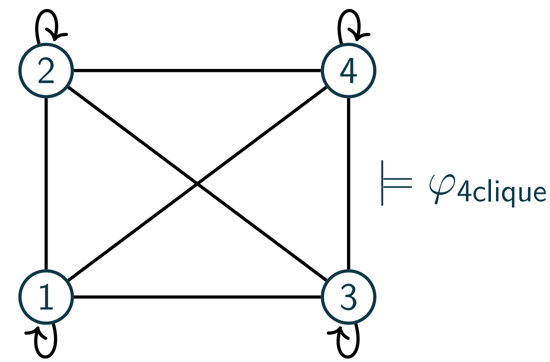

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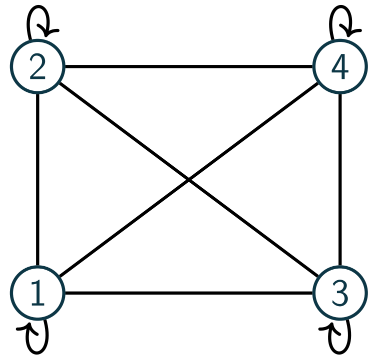

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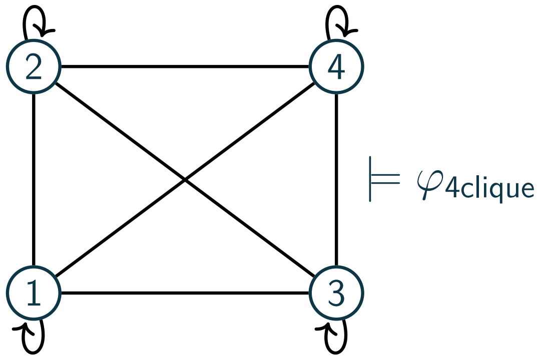

Exercise (Write a formula over $\{\mathrm{E}^{(2)}\}$ checking if a graph is two-colorable.)

More examples I.

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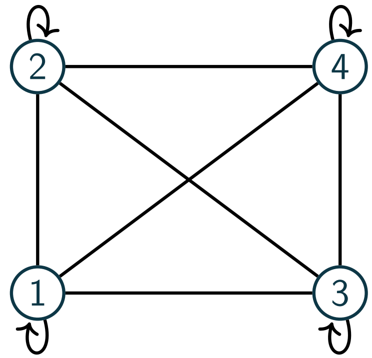

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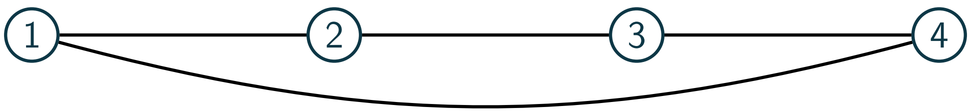

$$\varphi_{2COL} = \exists \mathrm{G} \exists \mathrm{R}\ (x \in \mathrm{G} \vee x \in \mathrm{R}) \wedge (x \in \mathrm{G} \leftrightarrow x \notin \mathrm{R})\ \wedge\ \varphi_{ok}$$

More examples I.

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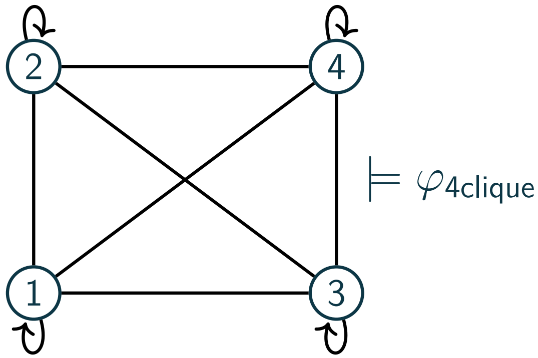

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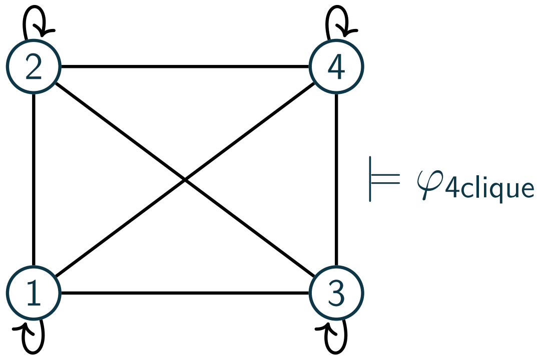

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Quantification over sets:

More examples I.

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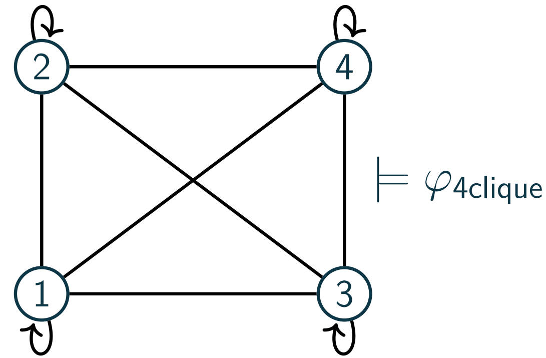

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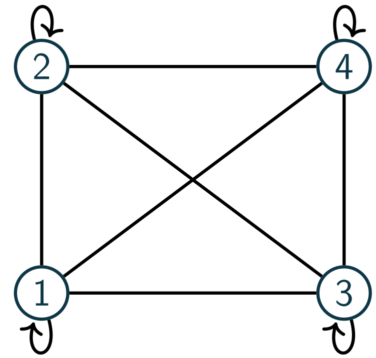

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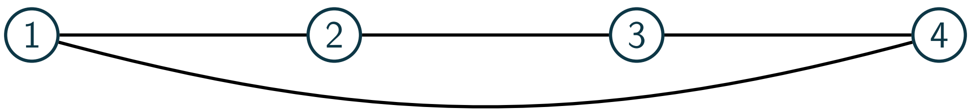
$\mathfrak{G} :=$ 

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There exists a colouring with G and R • and it is correct •

More examples I.

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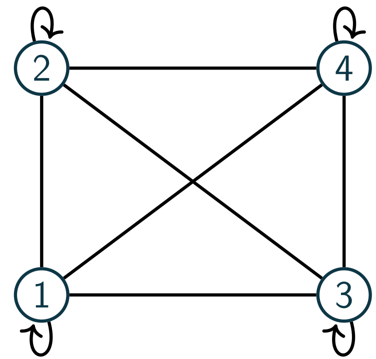

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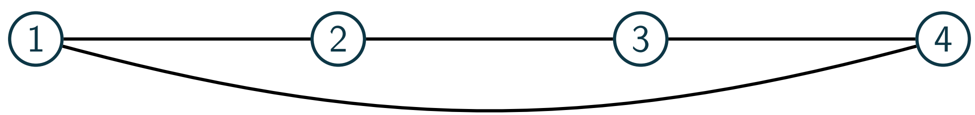
$\mathfrak{G} :=$ 

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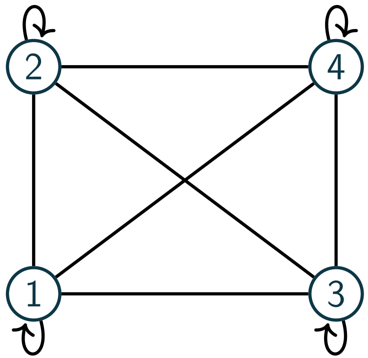

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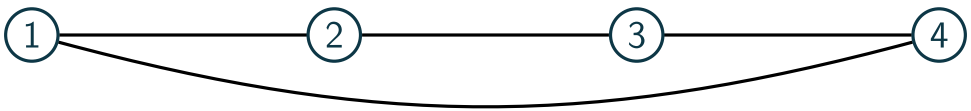

$\models \varphi_{2COL}$

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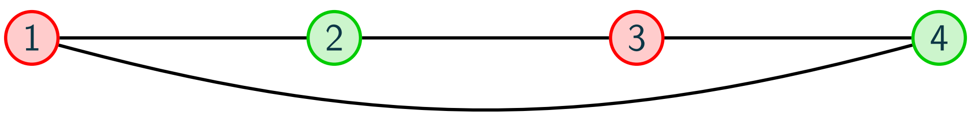

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More examples II.

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Exercise (Write an FO$[\{\mathrm{E}^{(2)}, \mathtt{a}, \mathtt{b}\}]$ formula $\varphi_k^{\mathsf{reach}(\mathtt{a},\mathtt{b})}$ testing if there is a path from $\mathtt{a}$ to $\mathtt{b}$ of length k.)

More examples II.

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1. Case $k = 0$ is trivial:

More examples II.

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More examples II.

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No. And we will show it today!

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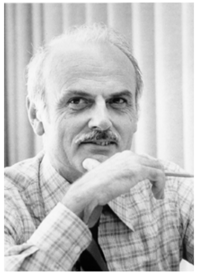

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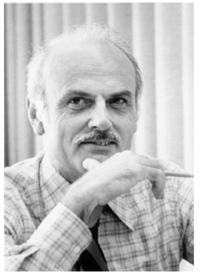

Other useful logic: Datalog ≈ SQL + recursion

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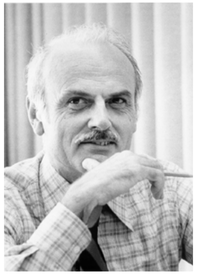

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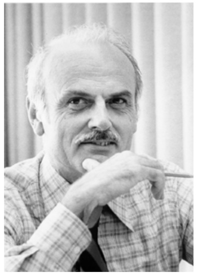

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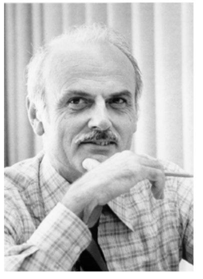

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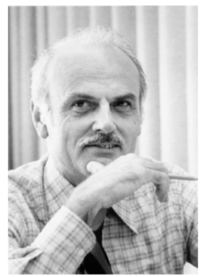

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Description logics: a family of logics for knowledge representation.

W3C® SNOMED CT The global language of healthcare protégé Dublin Core Metadata Initiative Making it easier to find information

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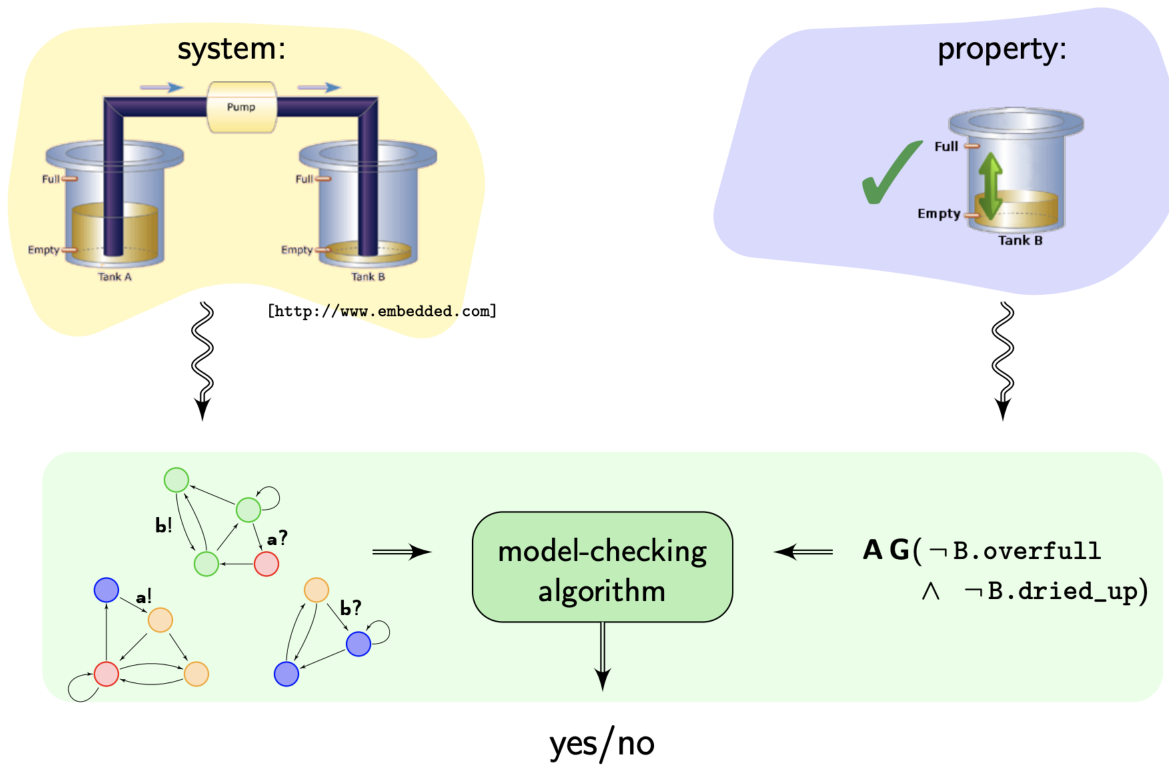

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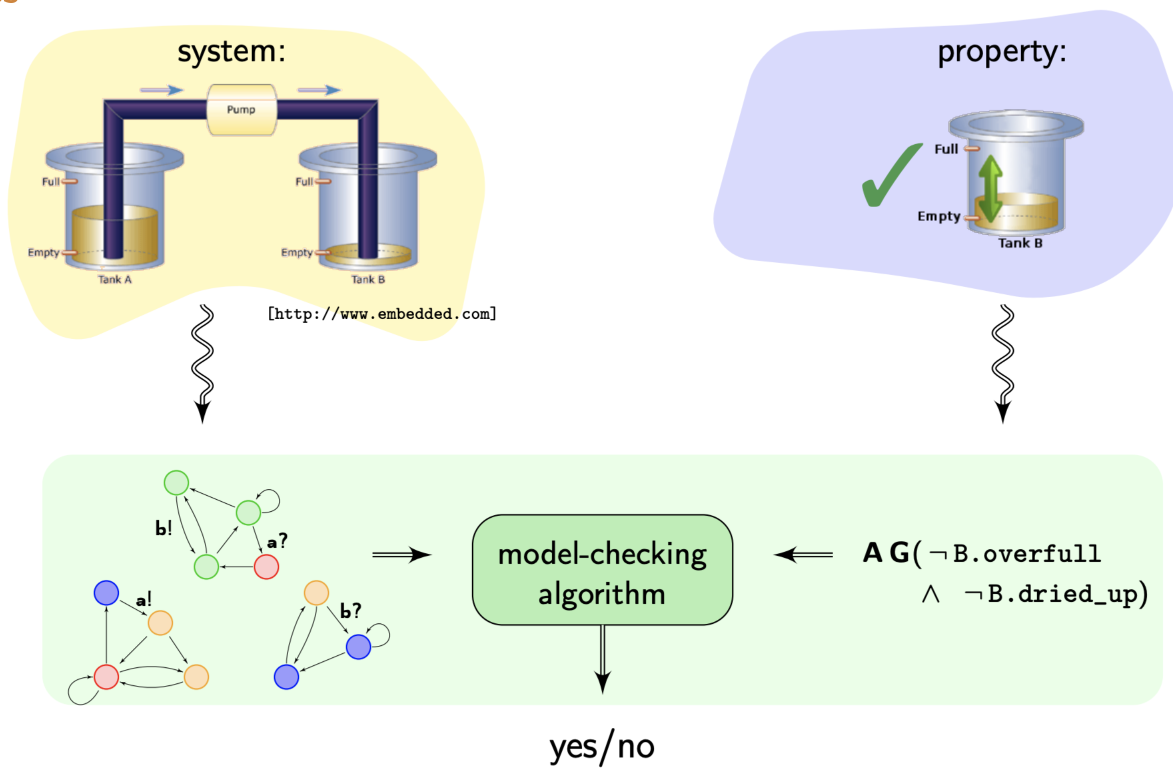

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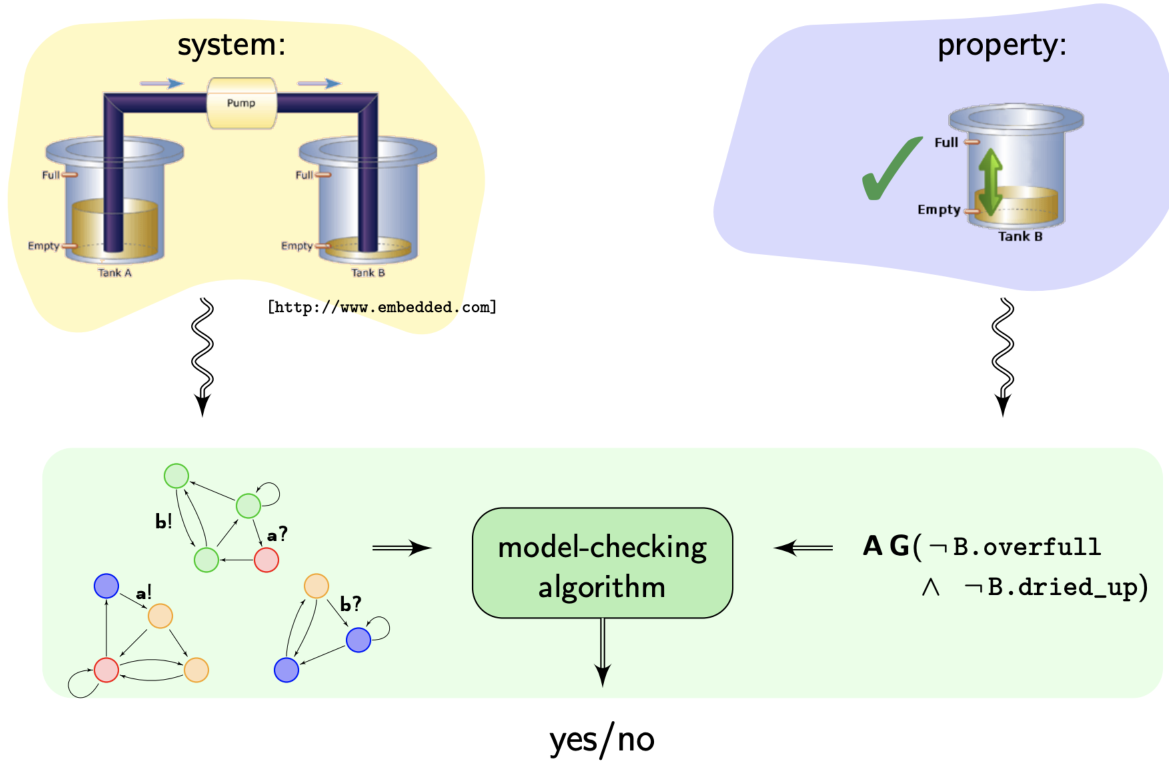

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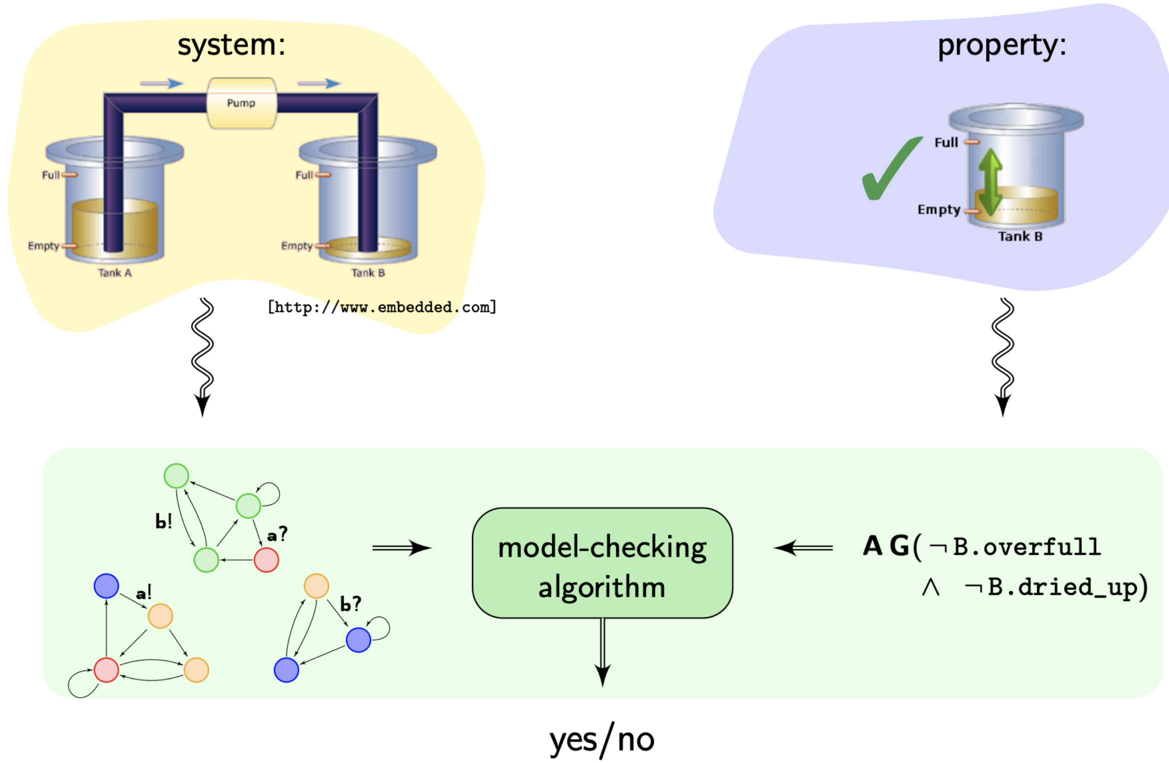

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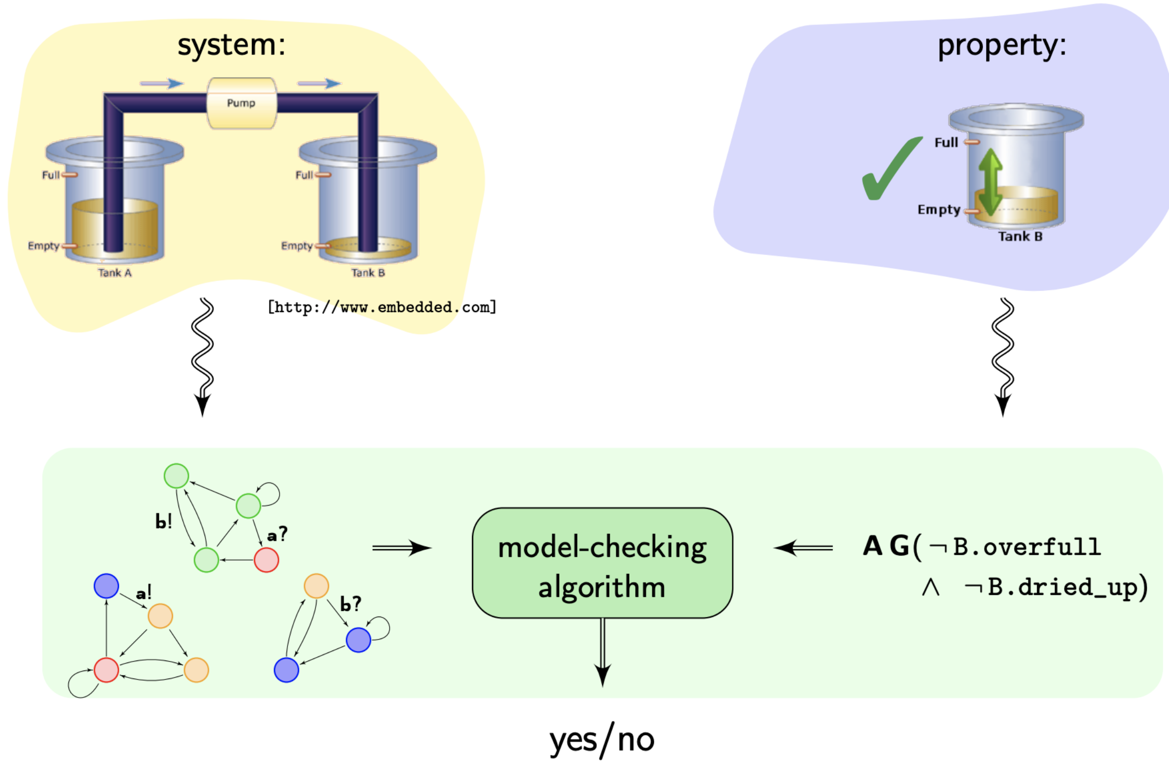

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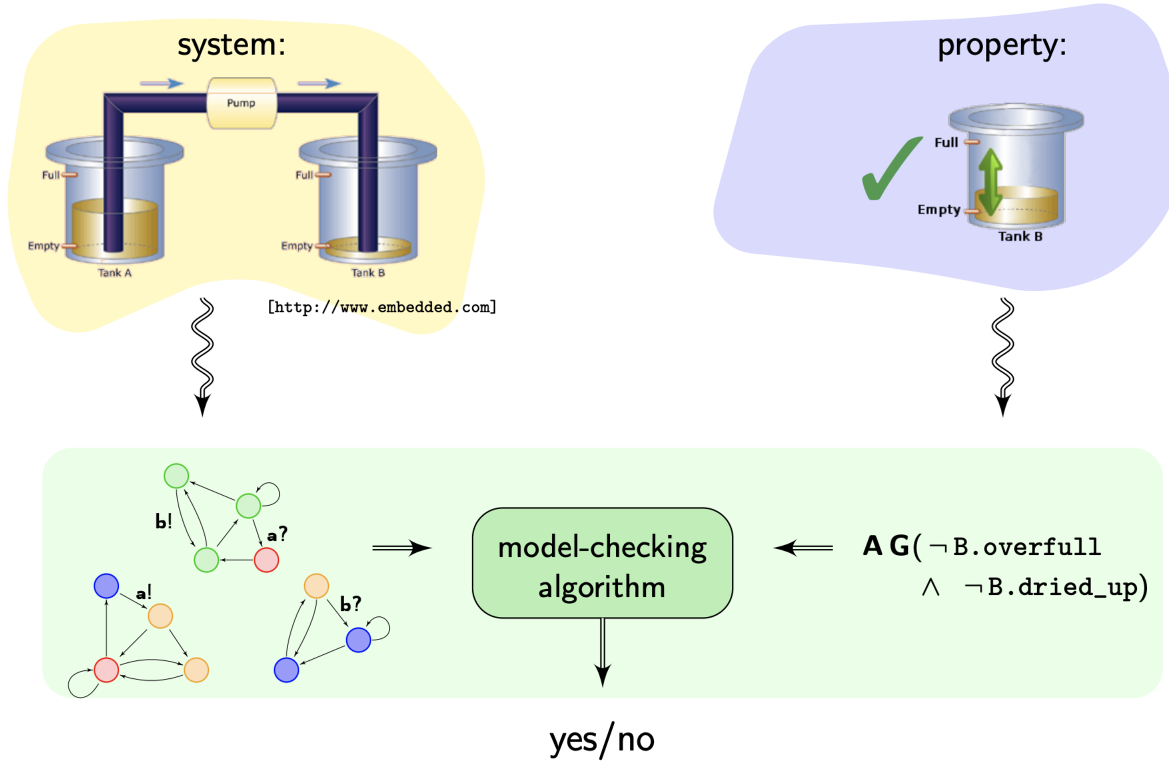

Motivations II: why do we care about logic?

1. Temporal logics as specification languages
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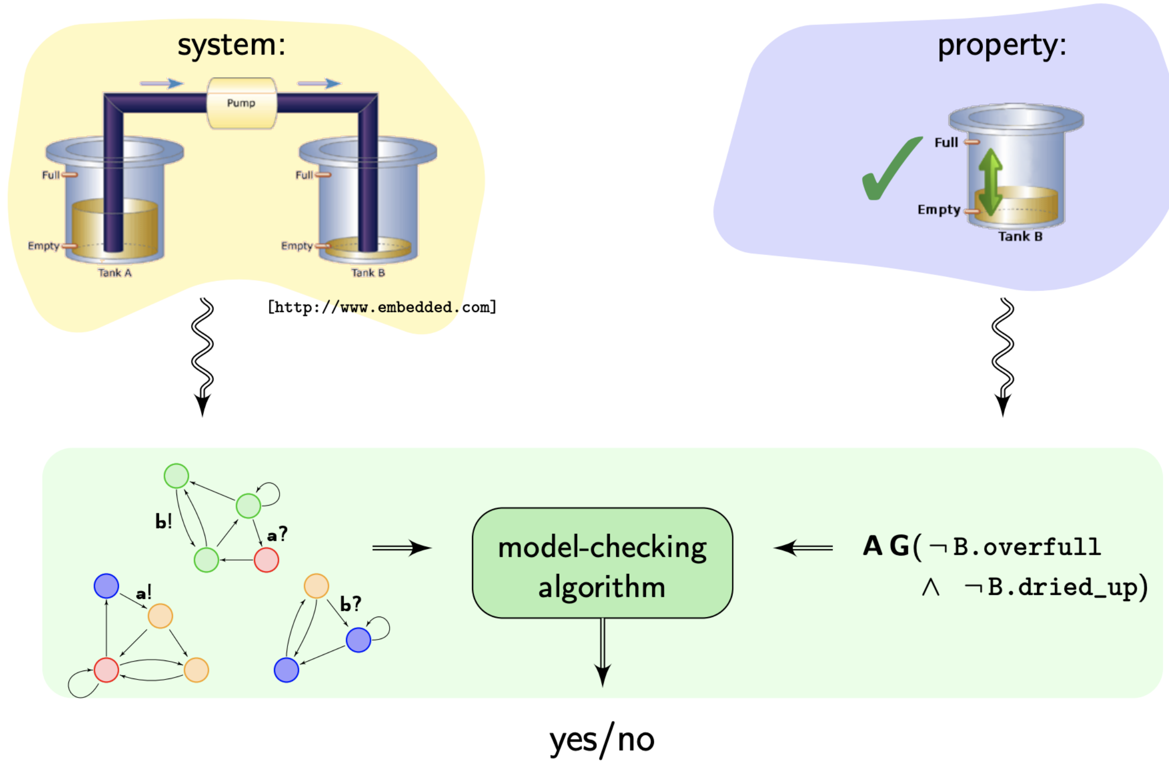

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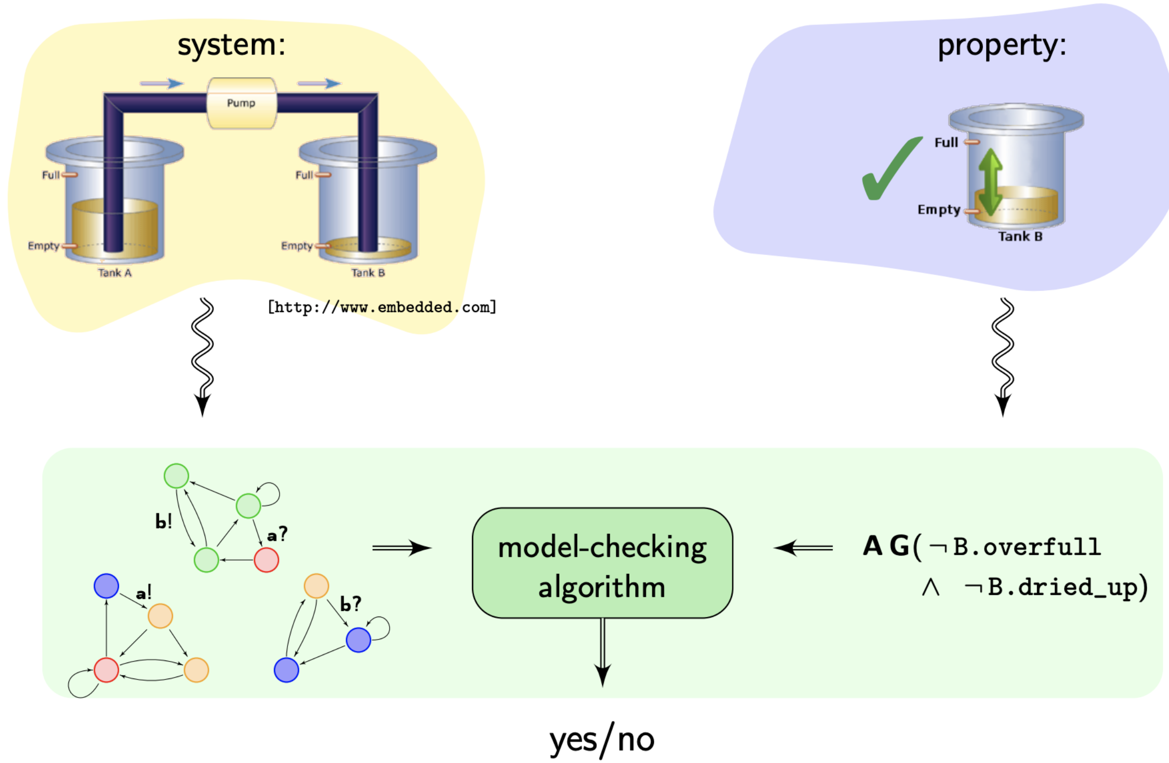

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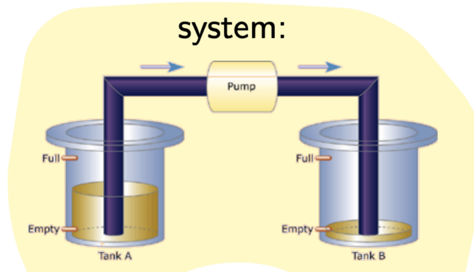

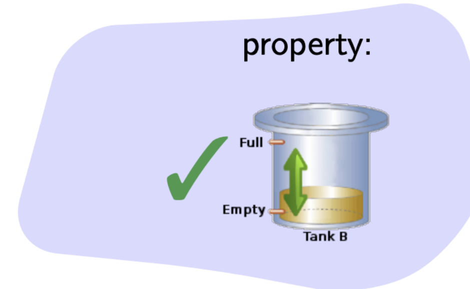

vim hello.c

```c
// hello.c
#include <stdlib.h>

void test() {
  int *s = NULL;
  *s = 42;
}
```

bartoszbednarczyk@Minsky-Machine: ~/Downloads/Infer

```
$ infer run -- gcc -c hello.c

Capturing in make/cc mode...
Found 1 source file to analyze in /Users/bartoszbednarczyk/Downloads/Infer/infer-out


Analysis finished in 775mss

Found 1 issue

hello.c:6: error: NULL_DEREFERENCE
  pointer `s` last assigned on line 5 could be null and is dereferenced at line 6, column 3.
  4.     void test() {
  5.       int *s = NULL;
  6. >     *s = 42;
  7.     }


Summary of the reports

  NULL_DEREFERENCE: 1
```

Motivations III: why do we care about logic?

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In “standard” computational complexity we measure resources, e.g. space and time.

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$O(n)$ time

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$\Theta(n \log(n))$ memory?

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solvable in PSpace?

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decidable?

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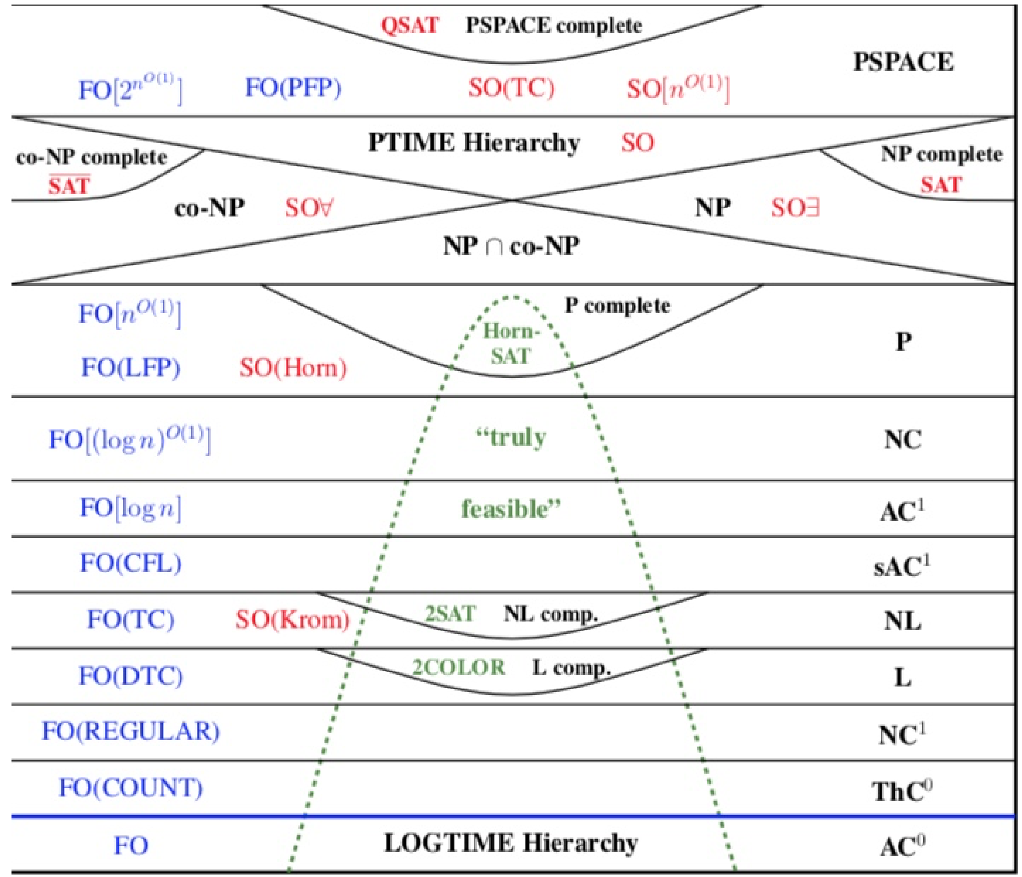

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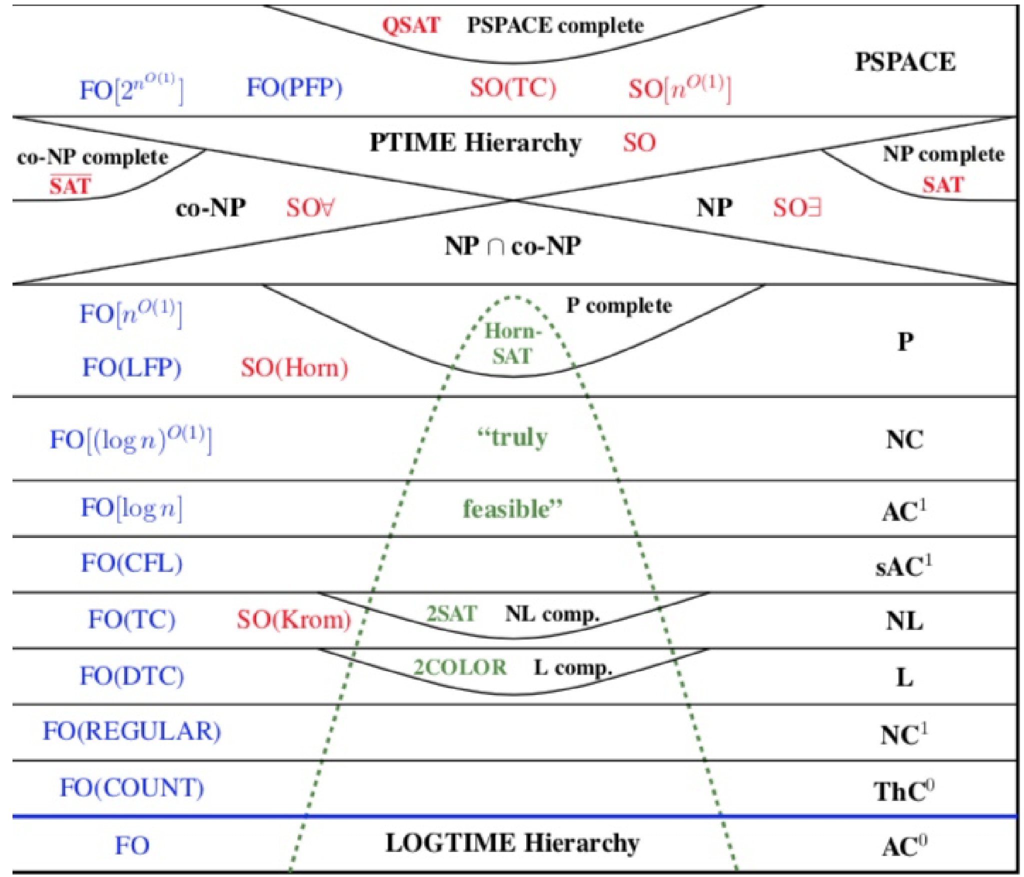

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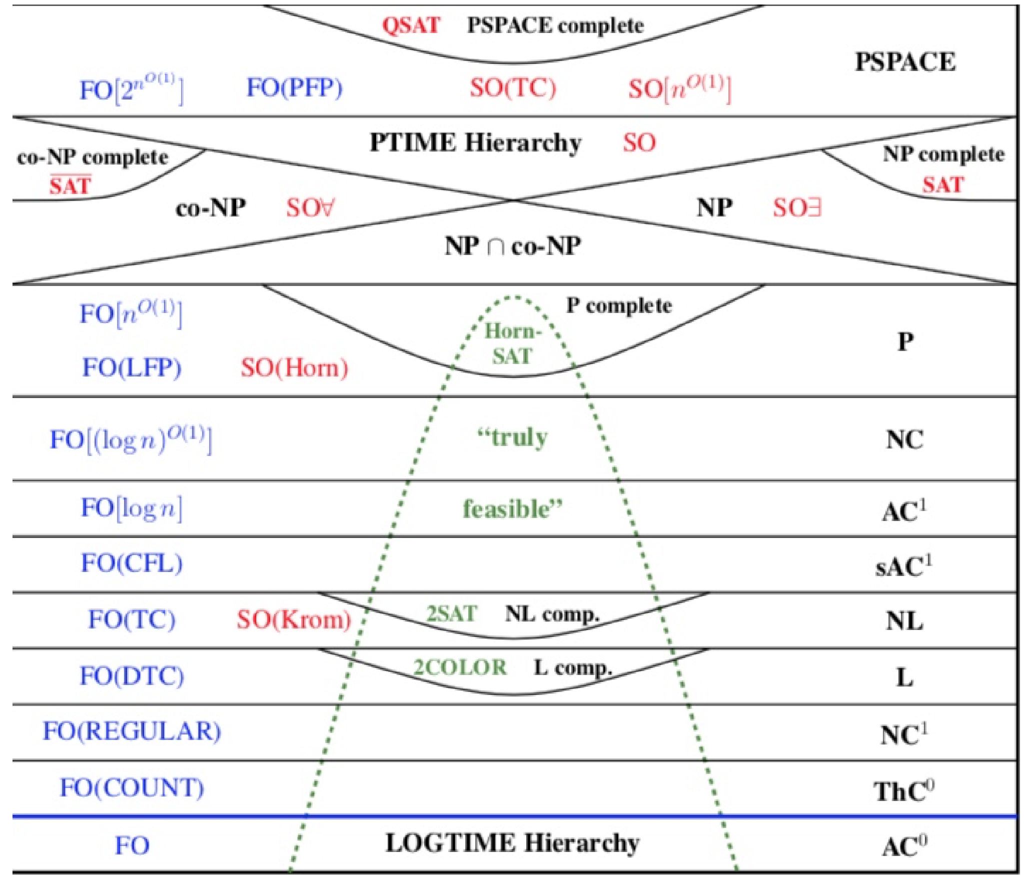

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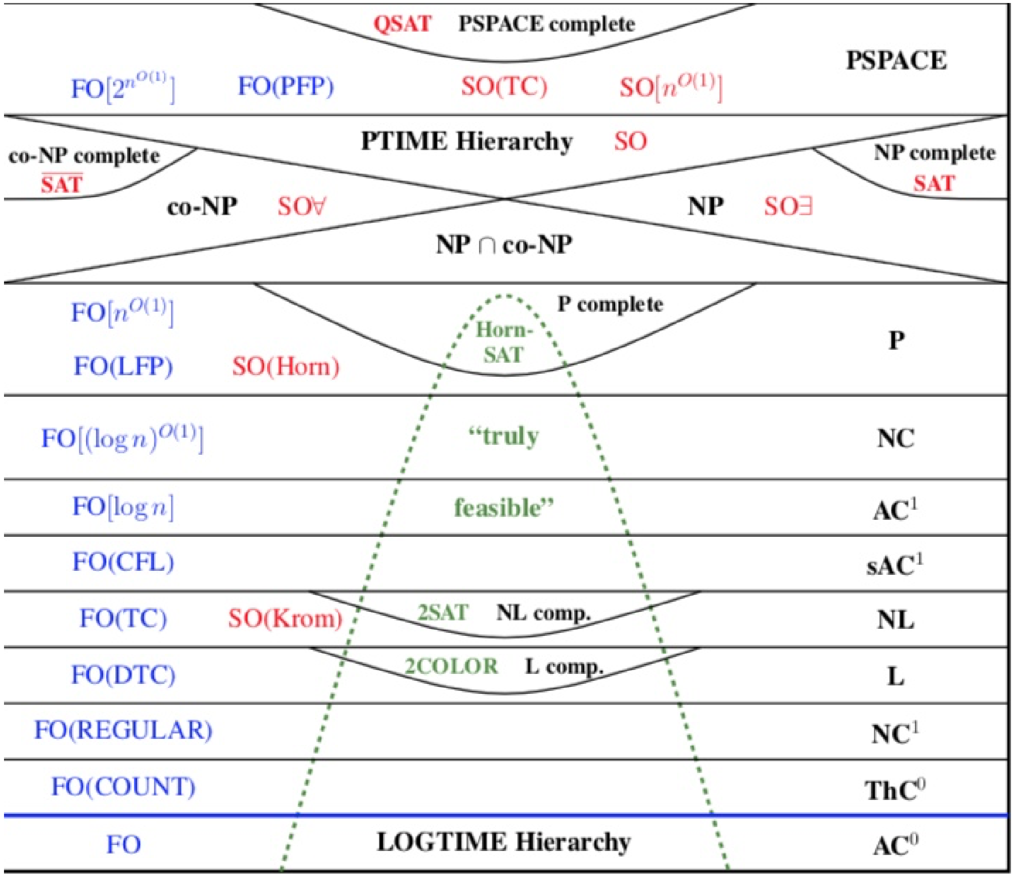

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Theorem (Fagin'1973)

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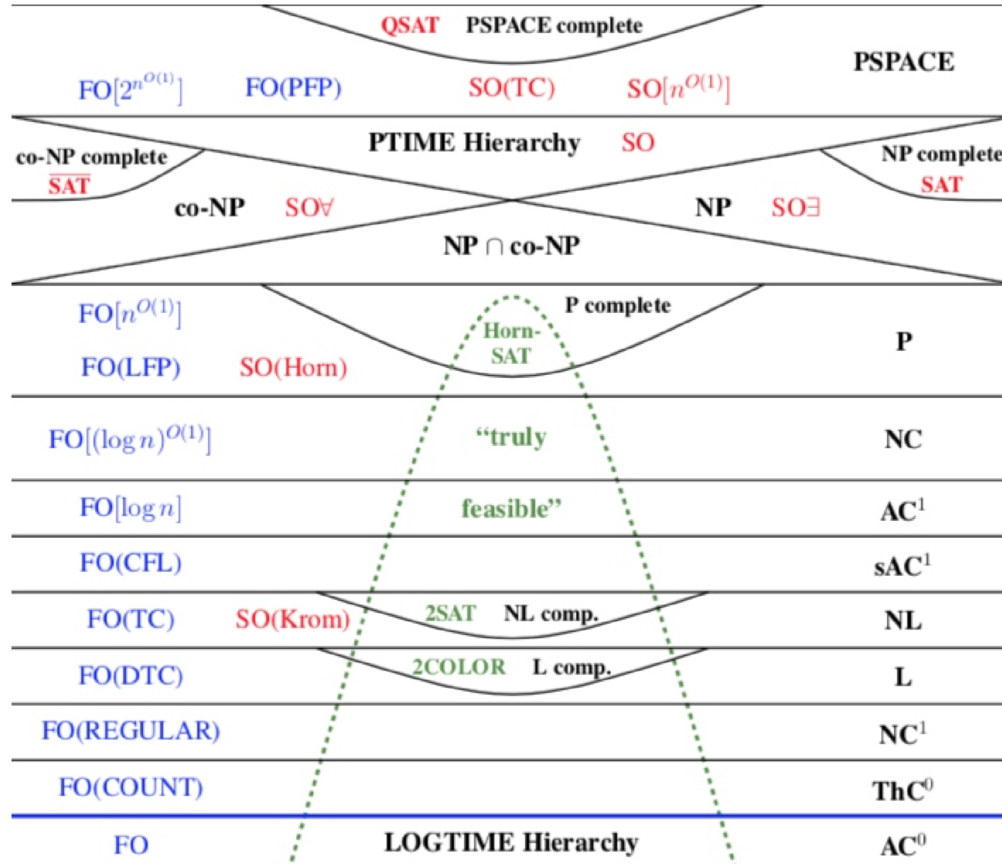

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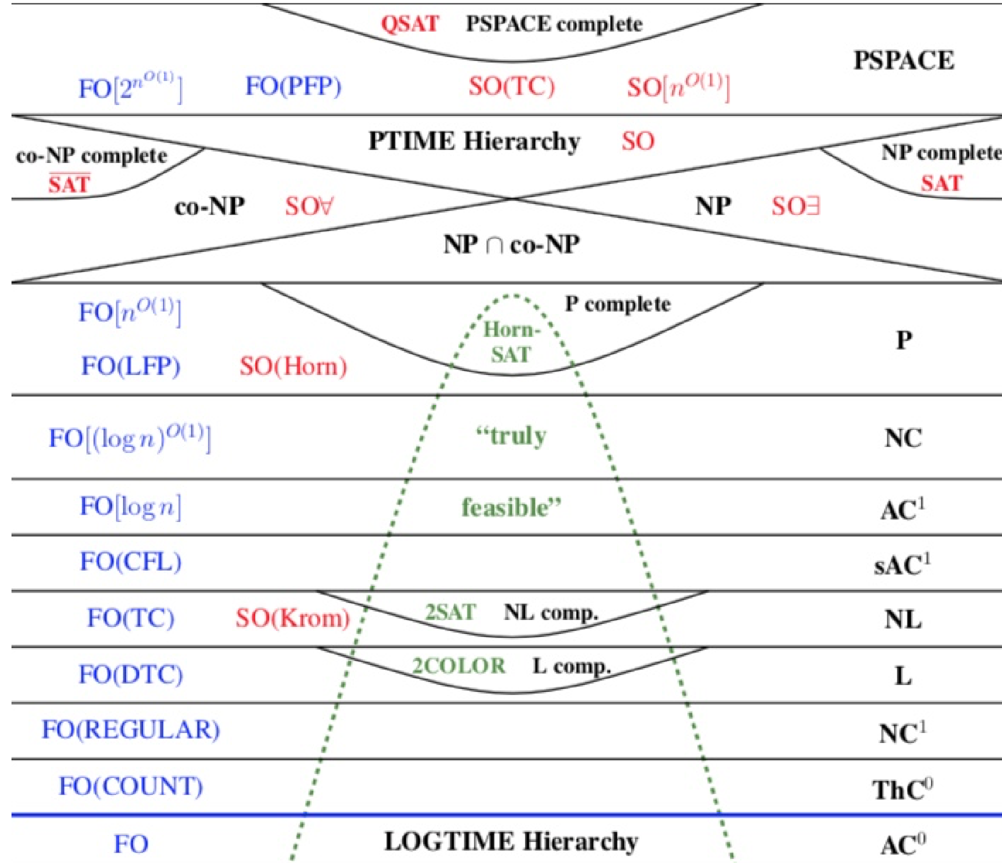

Theorem (Fagin'1973)

Existential Second Order Logic characterises NP.

Is there a logic for PTIME?

No idea since 1988.

Motivations IV: why do we care about logic?

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Meta algorithms: say what you want instead of writing a code! Hot topic nowadays!

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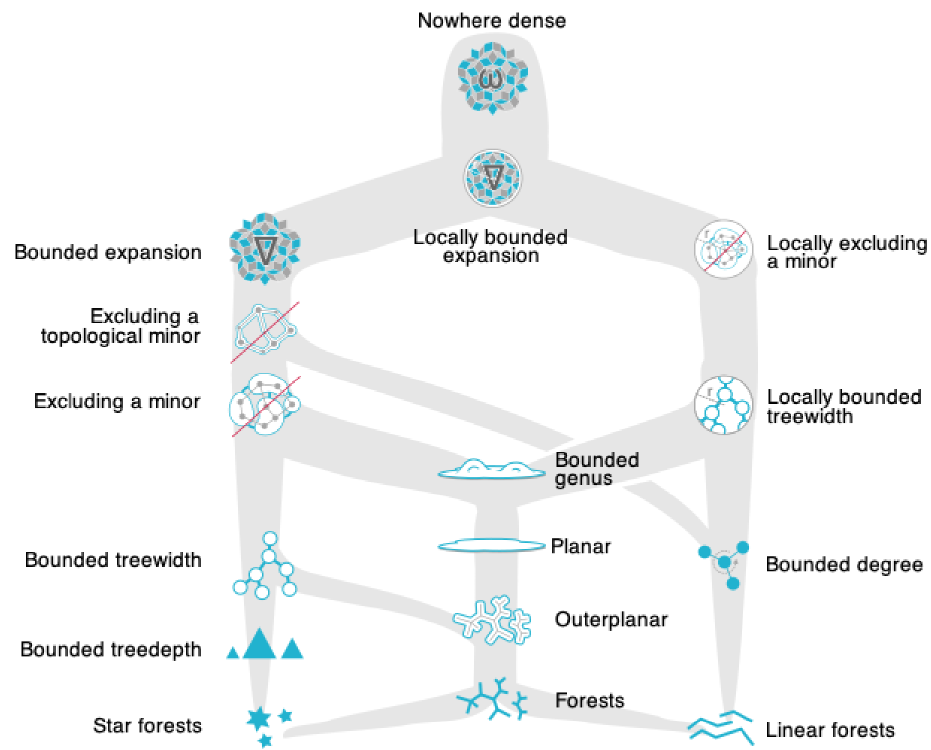

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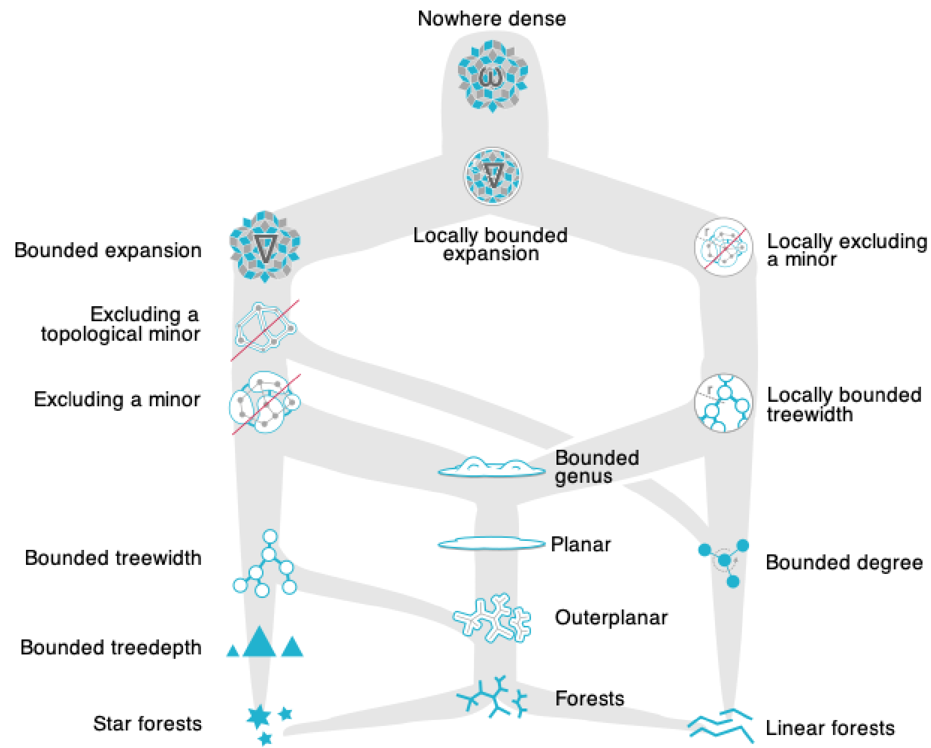
Theorem (Courcelle 1990)

$\mathcal{C} :=$ graphs of bounded-treewidth.



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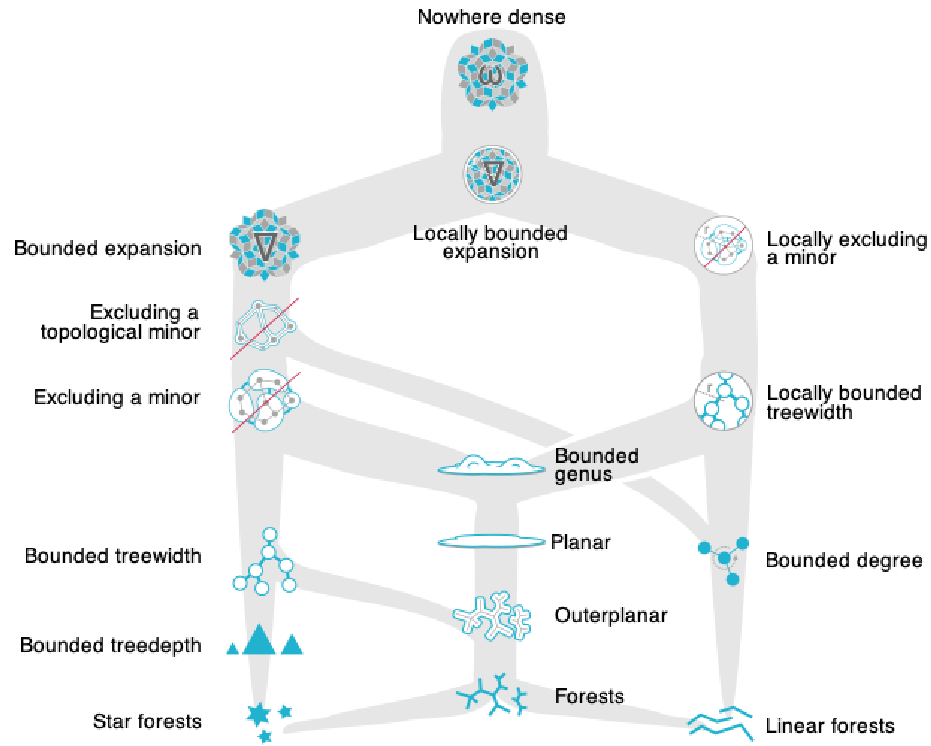

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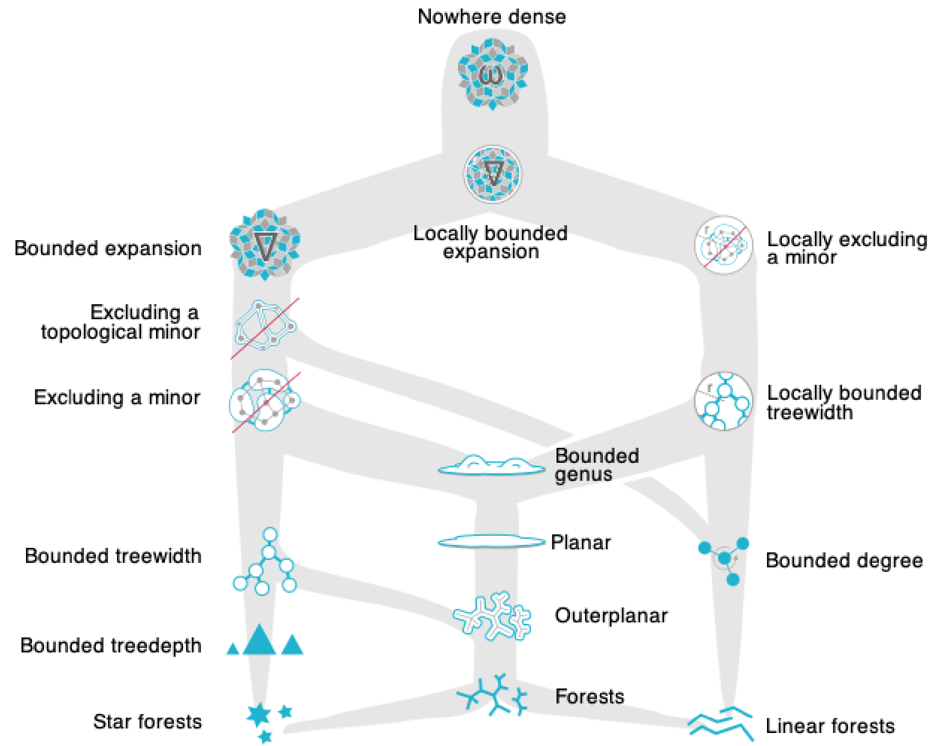

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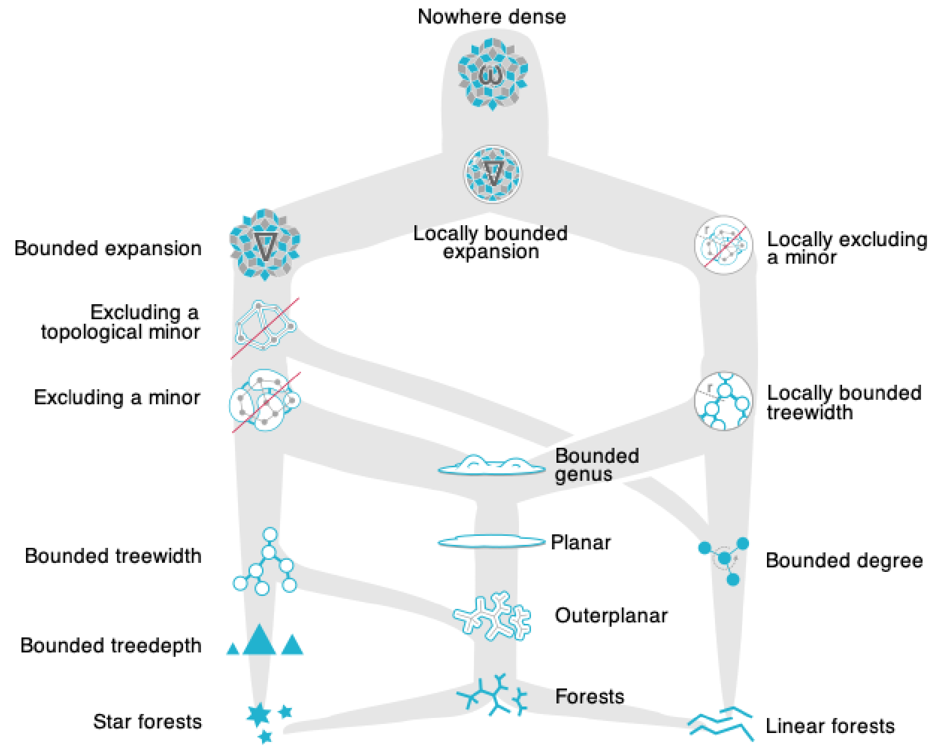

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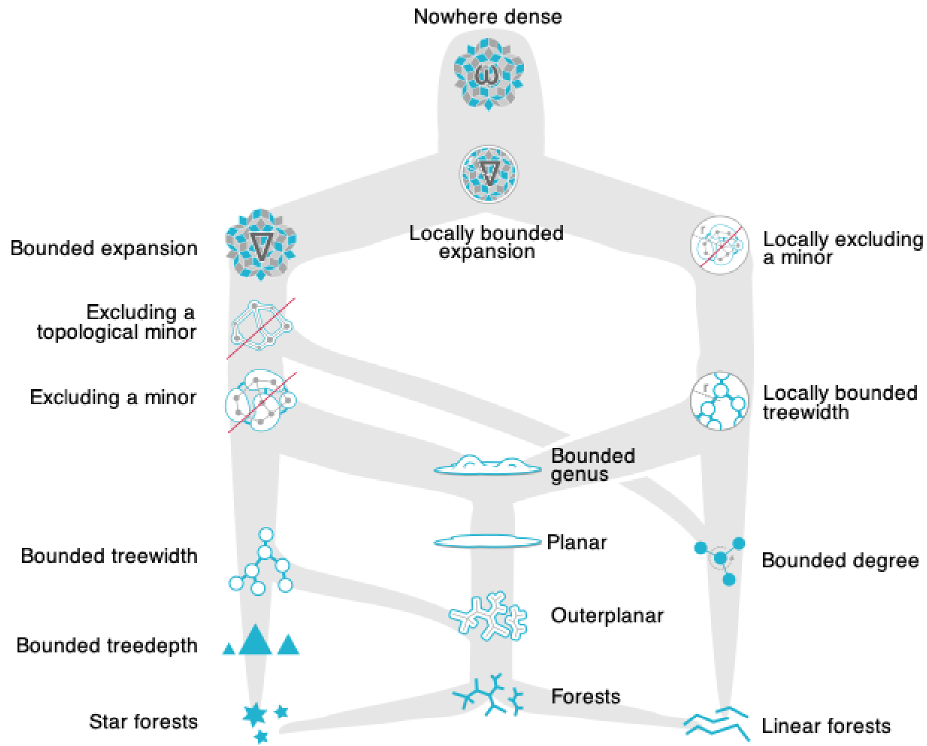
Theorem (Grohe, Kreutzer, Siebertz 2014)

$\mathcal{O}(|\varphi|^{1+\varepsilon})$ for $\mathcal{C}$:= nowhere-dense graphs.



4–5. Recap from BSc studies & Gödel's Completeness theorem

Signatures (vocabularies)

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Signature σ is a (countable) collection of symbols: $(\mathsf{c}_1, \mathsf{c}_2, \ldots, \mathrm{R}_1, \mathrm{R}_2, \ldots)$

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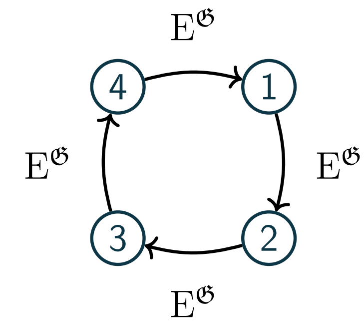

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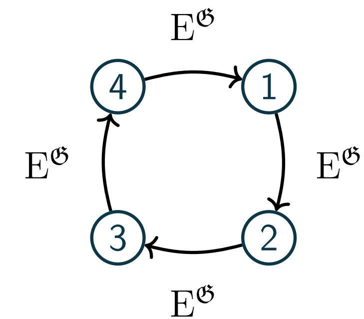

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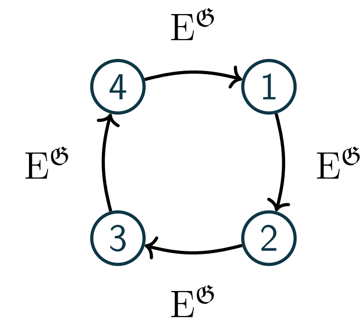

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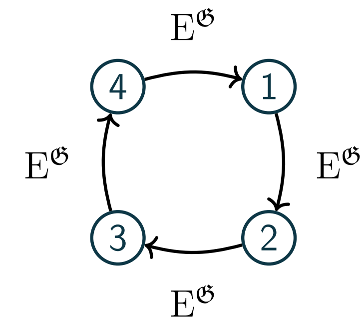

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Let $\mathfrak{A}, \mathfrak{B}$ be σ-structures. A homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ is $\mathfrak{h} : A \to B$ satisfying:

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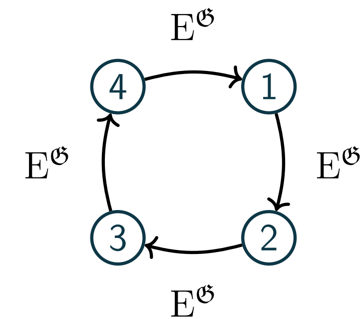

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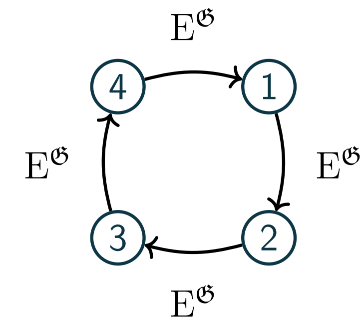

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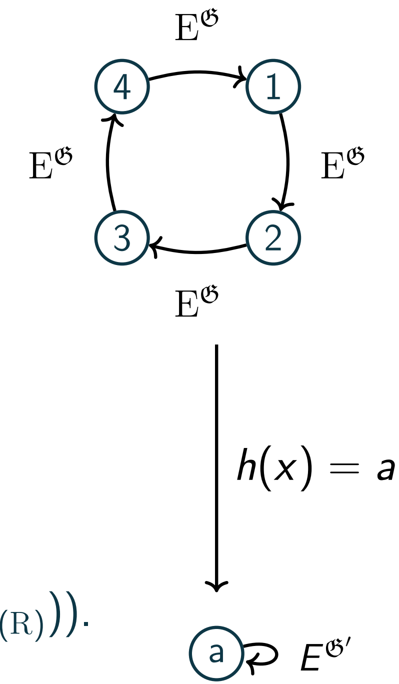

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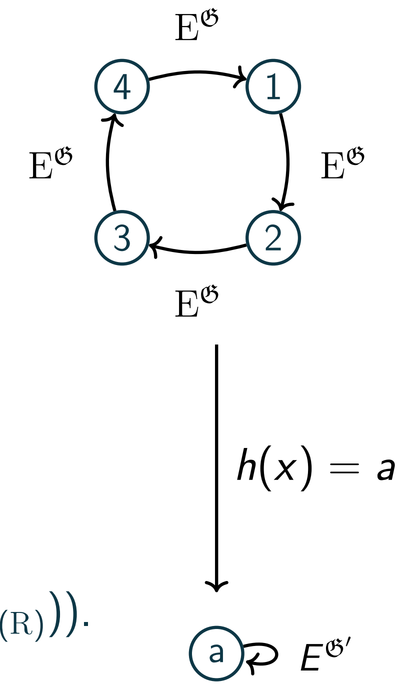

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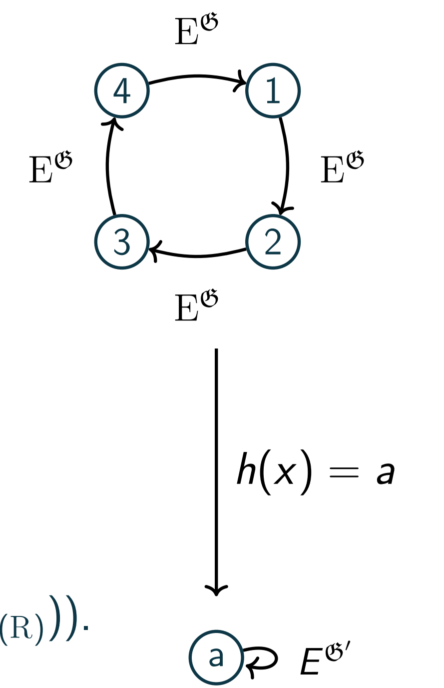

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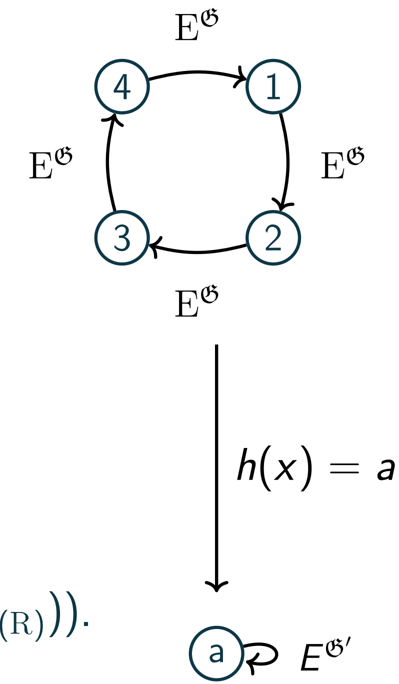

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Important! $\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi$ for all formulae φ.

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- If $\varphi \equiv \exists x\ \psi(x, \bar{y})$, then $\mathfrak{A} \models \varphi(\bar{a})$ iff $\mathfrak{A} \models \psi(a', \bar{a})$ for some $a' \in A$ (similarly for $\forall$ quantifier)

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SAT for FO is Recursively Enumerable

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4. The actual lecture

The Gödel's Compactness Theorem

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1. If $\mathcal{T} \models \varphi$ then there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ such that $\mathcal{T}_0 \models \varphi$.

The Gödel's Compactness Theorem

Let $\mathcal{T}$ be an FO-theory and let φ be an FO sentence.

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Use case:

Showing

inexpressivity

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1st excursion: Proving (1)

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"$\models \; = \; \vdash$"

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Assume $\mathcal{T} \models \varphi$. Then by Gödel's completeness theorem $\mathcal{T} \vdash \varphi$.

So there is a formal proof $\mathcal{P}$ of $\mathcal{T} \vdash \varphi$.

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Craft $\mathcal{T}_0$

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2nd excursion: Proving (2)

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Use case: Showing inexpressivity

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Ad absurdum

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Use case: Showing inexpressivity

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Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable.

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Ad absurdum

$\mathcal{T}$ unSAT iff $\mathcal{T} \models \bot$

2nd excursion: Proving (2)

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Ad absurdum

$\mathcal{T}$ unSAT iff $\mathcal{T} \models \perp$

Employ (1)

2nd excursion: Proving (2)

Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable. So $\mathcal{T} \models \perp$.

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Ad absurdum

$\mathcal{T}$ unSAT iff $\mathcal{T} \models \bot$

Employ (1)

2nd excursion: Proving (2)

Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable. So $\mathcal{T} \models \bot$.

By (1) there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ s.t. $\mathcal{T}_0 \models \bot$.

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Ad absurdum

$\mathcal{T}$ unSAT iff $\mathcal{T} \models \bot$

Employ (1)

2nd excursion: Proving (2)

Towards a contradiction suppose $\mathcal{T}$ is unsatisfiable. So $\mathcal{T} \models \bot$. By (1) there is a finite $\mathcal{T}_0 \subseteq \mathcal{T}$ s.t. $\mathcal{T}_0 \models \bot$. A contradiction!

Employing compactness I: Reachability in $\{E\}$-structures

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The general proof scheme to show that the property $\mathcal{P}$ is not FO-definable.

Employing compactness I: Reachability in $\{\mathrm{E}\}$-structures

The general proof scheme to show that the property $\mathcal{P}$ is not FO-definable.

Ad absurdum suppose that φ defines $\mathcal{P}$.

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The general proof scheme to show that the property $\mathcal{P}$ is not FO-definable.

Ad absurdum suppose that φ defines $\mathcal{P}$. $\rightsquigarrow$ Manufacture a theory $\mathcal{T}$ containing φ. $\rightsquigarrow$

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There is no FO[$\{\mathrm{E}\}$] formula for connectivity over $\{\mathrm{E}\}$-structures.

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There is no FO[$\{\mathrm{E}\}$] formula for connectivity over $\{\mathrm{E}\}$-structures.

So there is no formula saying that between any two nodes there is a directed $\{\mathrm{E}\}$-path.

Proof:

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$$\varphi_0^{\mathsf{reach(a,b)}} := \mathsf{a} = \mathsf{b},\ \varphi_1^{\mathsf{reach(a,b)}} := \mathrm{E}(\mathsf{a}, \mathsf{b}),\ \varphi_k^{\mathsf{reach(a,b)}} := \exists x_1 \ldots \exists x_{k-1}\ \mathrm{E}(\mathsf{a}, x_1) \land \land_{i=1}^{k-2} \mathrm{E}(x_i, x_{i+1}) \land \mathrm{E}(x_{k-1}, \mathsf{b})$$

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Assume that there is such φ, and let $\mathcal{T}$ be

$$\mathcal{T} := \{\varphi\} \cup \{\neg\varphi_k^{\mathrm{reach(a,b)}} \mid k \geq 0\}.$$

Employ reachability!

$\varphi_0^{\mathrm{reach(a,b)}} := \mathrm{a} = \mathrm{b}$, $\varphi_1^{\mathrm{reach(a,b)}} := \mathrm{E}(\mathrm{a},\mathrm{b})$, $\varphi_k^{\mathrm{reach(a,b)}} :=$

$\exists x_1 \ldots \exists x_{k-1}\, \mathrm{E}(\mathrm{a}, x_1) \wedge \wedge_{i=1}^{k-2} \mathrm{E}(x_i, x_{i+1}) \wedge \mathrm{E}(x_{k-1}, \mathrm{b})$

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Proof:

Assume that there is such φ, and let $\mathcal{T}$ be

$$\mathcal{T} := \{\varphi\} \cup \{\neg\varphi_k^{\mathrm{reach(a,b)}} \mid k \geq 0\}.$$

Since a and b are disconnected, $\mathcal{T}$ is unSAT.

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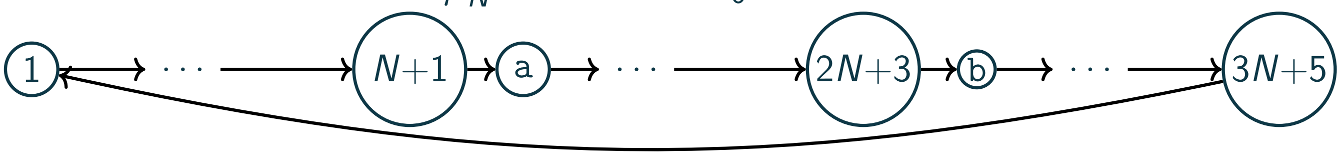

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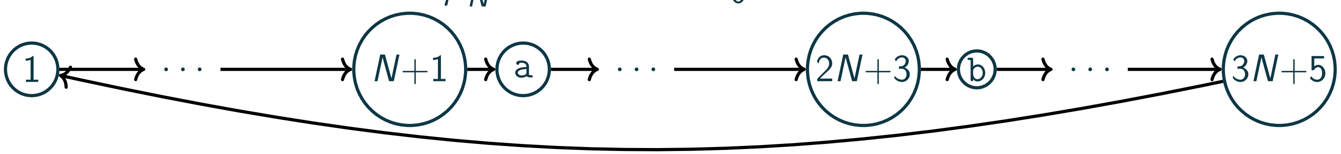

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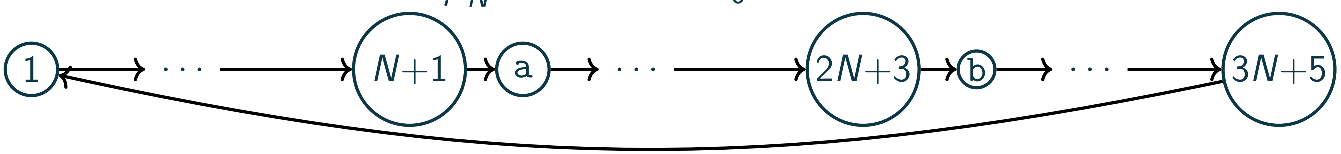

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