Technische Universität Dresden

International Center for Computational Logic

COMPLEXITY THEORY

Lecture 2: Turing Machines and Languages

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Knowledge-Based Systems

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Looking for Project or Thesis Topics?

On **Thursday, 16 Oct 2025 at 2.50pm in APB 3027** we will present possible topics to conduct in the **Knowledge-Based Systems** group as a **study project** (many suitable modules) or **final thesis** (BSc, MSc, Diploma).

Not only **theoretical topics** but also **implementation work**.

We also have student **job opportunities (SHK/WHK)**.

You are especially welcome if you are eager to work with **Rust** or **LEAN** :)

See also: `https://iccl.inf.tu-dresden.de/web/Projekte_und_Studienarbeiten_Wissensbasierte_Systeme/en`

A Model for Computation

Clear

To understand computational problems we need to have a formal understanding of what an **algorithm** is.

Example 2.1 (Hilbert's Tenth Problem):

> "Given a Diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers."
> ($\rightarrow$ Wikipedia)

Question

How can we model the notion of an algorithm?

Answer

With Turing machines.

Turing Machines

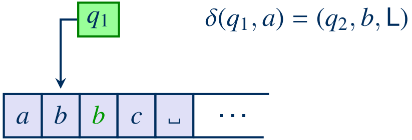
Let us fix a blank symbol $\sqcup$.

Definition 2.2: A (deterministic) Turing Machine $\mathcal{M} = \langle Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}} \rangle$ consists of

- a finite set Q of states,
- an input alphabet Σ not containing $\sqcup$,
- a tape alphabet Γ such that $\Gamma \supseteq \Sigma \cup \{\sqcup\}$.
- a transition function $\delta \colon Q \times \Gamma \to Q \times \Gamma \times \{\mathsf{L}, \mathsf{R}\}$
- an initial state $q_0 \in Q$,
- an accepting state $q_{\text{accept}} \in Q$, and
- a rejecting state $q_{\text{reject}} \in Q$ such that $q_{\text{accept}} \neq q_{\text{reject}}$.

Turing Machines

Example 2.3:



- The tape is bounded on the left, but unbounded on the right; the content of the tape is a finite word over Γ, followed by an infinite sequence of $\textvisiblespace$.
- The head of the machine is at exactly one position of the tape
- The head can read only one symbol at a time
- The head moves and writes according to the transition function δ; the current state also changes accordingly
- The head will stay put when attempting to cross the left tape end

Configurations

Observation: to describe the current step of a computation of a TM it is enough to know

- the content of the tape,
- the current state, and
- the position of the head

Definition 2.4: A configuration of a TM $\mathcal{M}$ is a word uqv such that

- $q \in Q$,
- $uv \in \Gamma^*$

Some special configurations:

- The **start configuration** for some input word $w \in \Sigma^*$ is the configuration $q_0 w$
- A configuration uqv is **accepting** if $q = q_{\mathsf{accept}}$.
- A configuration uqv is **rejecting** if $q = q_{\mathsf{reject}}$.

Computation

We write

- $C \vdash_{\mathcal{M}} C'$ only if C' can be reached from C by one computation step of $\mathcal{M}$;
- $C \vdash^{*}_{\mathcal{M}} C'$ only if C' can be reached from C in a finite number of computation steps of $\mathcal{M}$.

We say that $\mathcal{M}$ **halts** on input w if and only if there is a finite sequence of configurations

$$C_0 \vdash_{\mathcal{M}} C_1 \vdash_{\mathcal{M}} \cdots \vdash_{\mathcal{M}} C_\ell$$

such that C_0 is the start configuration of $\mathcal{M}$ on input w and C_ℓ is an accepting or rejecting configuration. Otherwise $\mathcal{M}$ **loops** on input w.

We say that $\mathcal{M}$ **accepts** the input w only if $\mathcal{M}$ halts on input w with an accepting configuration.

Recognisability and Decidability

Definition 2.5: Let $\mathcal{M}$ be a Turing machine with input alphabet Σ. The language accepted by $\mathcal{M}$ is the set

$$\mathbf{L}(\mathcal{M}) := \{ w \in \Sigma^* \mid \mathcal{M} \text{ accepts } w \}.$$

A language $\mathbf{L} \subseteq \Sigma^*$ is called Turing-recognisable (recursively enumerable) if and only if there exists a Turing machine $\mathcal{M}$ with input alphabet Σ such that $\mathbf{L} = \mathbf{L}(\mathcal{M})$. In this case we say that $\mathcal{M}$ recognises $\mathbf{L}$.

A language $\mathbf{L} \subseteq \Sigma^*$ is called Turing-decidable (decidable, recursive) if and only if there exists a Turing machine $\mathcal{M}$ such that $\mathbf{L} = \mathbf{L}(\mathcal{M})$ and $\mathcal{M}$ halts on every input. In this case we say that $\mathcal{M}$ decides $\mathbf{L}$.

Example

Claim 2.6: The language **L** $:= \{ \mathrm{a}^{2^n} \mid n \geq 0 \}$ is decidable.

Proof: A Turing machine $\mathcal{M}$ that decides **L** is

$\mathcal{M}$:= On input w, where w is a string

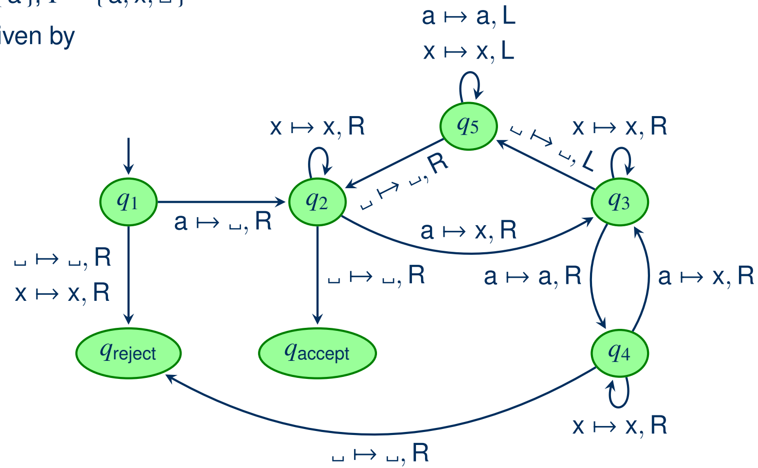
- Go from left to right over the tape and cross off every other a
- If in the first step the tape contained a single a, accept
- If in the first step the number of a's on the tape was odd, reject
- Return the head the beginning of the tape
- Go to the first step

Example (cont'd)

Formally, $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_1, q_{\text{accept}}, q_{\text{reject}})$, where

- $Q = \{ q_1, q_2, q_3, q_4, q_5, q_{\text{accept}}, q_{\text{reject}} \}$
- $\Sigma = \{ \mathsf{a} \}$, $\Gamma = \{ \mathsf{a}, \mathsf{x}, \sqcup \}$

and δ is given by



Problems as Languages

Observation

- Languages can be used to model computational problems.
- For this, a suitable **encoding** is necessary
- TMs must be able to decode the encoding

Example 2.7 (Graph-Connectedness): The question whether a graph is connected or not can be seen as the **word problem** of the following language

$$\text{GCONN} := \{\, \langle G \rangle \mid G \text{ is a connected graph} \,\},$$

where $\langle G \rangle$ is (for example) the adjacency matrix encoded in binary.

Notation 2.8: The encoding of objects $O_1, \ldots, O_n$ we denote by $\langle O_1, \ldots, O_n \rangle$.

The Church-Turing Thesis

It turns out that Turing-machines are **equivalent** to a number of formalisations of the intuitive notion of an **algorithm**

- λ-calculus
- while-programs
- μ-recursive functions
- Random-Access Machines
- $\ldots$

Because of this it is believed that Turing-machines completely capture the intuitive notion of an algorithm. $\rightsquigarrow$ **Church-Turing Thesis**:

> "A function on the natural numbers is intuitively computable if and only if it can be computed by a Turing machine."

(→ Wikipedia: Church-Turing Thesis)

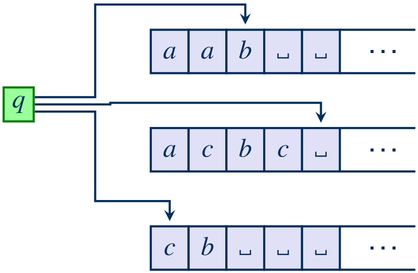
Variations of Turing-Machines

It has also been shown that deterministic, single-tape Turing machines are equivalent to a wide range of other forms of Turing machines:

- Multi-tape Turing machines
- Nondeterministic Turing machines
- Turing machines with doubly-infinite tape
- Multi-head Turing machines
- Two-dimensional Turing machines
- Write-once Turing machines
- Two-stack machines
- Two-counter machines
- ...

Multi-Tape Turing Machines

k-tape Turing machines are a variant of Turing machines that have k tapes.



Multi-Tape Turing Machines

Definition 2.9: Let $k \in \mathbb{N} \setminus \{0\}$. Then a (deterministic) k**-tape Turing machine** is a tuple $\mathcal{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where

- Q, Σ, Γ, q_0, q_{accept}, q_{reject} are as for TMs
- δ is a transition function for k tapes, i.e.,

$$\delta \colon Q \times \Gamma^k \to Q \times \Gamma^k \times \{\mathsf{L}, \mathsf{R}, \mathsf{N}\}^k$$

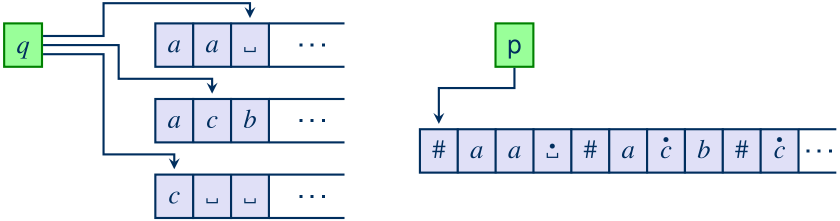
Running $\mathcal{M}$ on input $w \in \Sigma^*$ means to start $\mathcal{M}$ with the content of the first tape being w and all other tapes blank.

The notions of a **configuration** and of the **language accepted by** $\mathcal{M}$ are defined analogously to the single-tape case.

Theorem 2.10: Every multi-tape Turing machine has an equivalent single-tape Turing machine.

Proof: Let $\mathcal{M}$ be a k-tape Turing machine. Simulate $\mathcal{M}$ with a single-tape TM S by

- keeping the content of all k tapes on a single tape, separated by #
- marking the positions of the individual heads using special symbols



Multi-Tape Turing Machines

$S :=$ On input $w = w_1 \dots w_n$

- Format the tape to contain the word

$$\#\dot{w}_1 w_2 \dots w_n \# \dot{\sqcup} \# \dot{\sqcup} \# \dots \#$$

- Scan the tape from the first # to the $(k+1)$-th # to determine the symbols below the markers.
- Update all tapes according to $\mathcal{M}$'s transition function with a second pass over the tape; if any head of $\mathcal{M}$ moves to some previously unread portion of its tape, insert a blank symbol at the corresponding position and shift the right tape contents by one cell
- Repeat until the accepting or rejecting state is reached.

□

Nondeterministic Turing Machines

Goal

Allow transitions to be **nondeterministic**.

Approach

Change transition function from

$$\delta\colon Q \times \Gamma \to Q \times \Gamma \times \{\,\mathsf{L}, \mathsf{R}\,\}$$

to

$$\delta\colon Q \times \Gamma \to 2^{Q \times \Gamma \times \{\,\mathsf{L},\mathsf{R}\,\}}.$$

The notions of **accepting** and **rejecting computations** are defined accordingly.
Note: there may be more than one or no computation of a nondeterministic TM on a given input.
A nondeterministic TM $\mathcal{M}$ **accepts** an input w if and only if **there exists** some accepting computation of $\mathcal{M}$ on input w.

Theorem 2.11: Every nondeterministic TM has an equivalent deterministic TM.

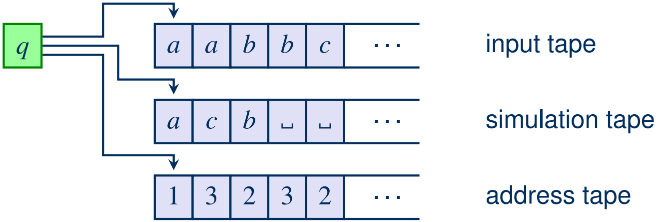
Proof: Let N be a nondeterministic TM. We construct a deterministic TM D that is equivalent to N, i.e., $\mathbf{L}(N) = \mathbf{L}(D)$.

Idea

- D deterministically traverses in breadth-first order the tree of configuration of N, where each branch represents a different possibility for N to continue.
- For this, successively try out all possible choices of transitions allowed by N.

Nondeterministic Turing Machines

Sketch of D:



Let b be the maximal number of choices in δ, i.e.,

$$b := \max\{\, |\delta(q, x)| \;\big|\; q \in Q, x \in \Gamma \,\}.$$

Nondeterministic Turing Machines

D works as follows:

(1) Start: input tape contains input w, simulation and address tape empty

(2) Initialise the address tape with 0.

(3) Copy w to the simulation tape.

(4) Simulate one finite computation of N on w on the simulation tape.
   - Interpret the address tape as a list of zero-indexed choices to make during this computation (and abort if the end of the tape is reached).
   - If a choice is invalid, abort simulation.
   - If an accepting configuration is reached at the end of the simulation, accept.

(5) "Increment" the content of the address tape by 1, intuitively considered as a number in base b **but** $b-1$ increments to 00, $0b-1$ to 10 and so on.
Go to step 3.

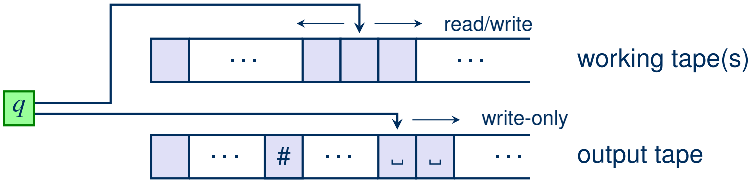
□

Enumerators

Definition 2.12: A multi-tape Turing machine $\mathcal{M}$ is an enumerator if

- $\mathcal{M}$ has a designated write-only output-tape on which a symbol, once written, can never be changed and where the head can never move left;
- $\mathcal{M}$ has a marker symbol # separating words on the output tape.

We define the language generated by $\mathcal{M}$ to be the set $\mathbf{G}(\mathcal{M})$ of all words that eventually appear between two consecutive # on the output tape of $\mathcal{M}$ when started on the empty word as input.



Enumerators

Theorem 2.13: A language **L** is Turing-recognisable if and only if there exists some enumerator $\mathcal{E}$ such that $\mathbf{G}(\mathcal{E}) = \mathbf{L}$.

Proof: Let $\mathcal{E}$ be an enumerator for **L**. Then the following TM accepts **L**:

$\mathcal{M}$:= On input w

- Simulate $\mathcal{E}$ on the empty input. Compare every string output by $\mathcal{E}$ with w
- If w appears in the output of $\mathcal{E}$, accept

Enumerators

Let **L** $= \mathbf{L}(\mathcal{M})$ for some TM $\mathcal{M}$, and let $s_1, s_2, \ldots$ be an enumeration of Σ^*.
Then the following enumerator $\mathcal{E}$ enumerates **L**:

$\mathcal{E} :=$ Ignore the input.

- Print the first # to initialise the output.
- Repeat for $i = 1, 2, 3, \ldots$
  - Run $\mathcal{M}$ for i steps on each input $s_1, s_2, \ldots, s_i$
  - If any computation accepts, print the corresponding s_j followed by #

□

Theorem 2.14: If **L** is Turing-recognisable, then there exists an enumerator for **L** that prints each word of **L** exactly once.

Enumerators

Theorem 2.15: A language **L** is decidable if and only if there exists an enumerator for **L** that outputs exactly the words of **L** in some order of non-decreasing length.

Proof: Suppose **L** to be decidable, and let $\mathcal{M}$ be a TM that decides **L**.

- Define a TM $\mathcal{M}'$ that generates, on some scratch tape, all words over Σ in some order of non-decreasing length. (Exercise!)
- An enumerator $\mathcal{E}$ works as follows:
  (1) Print the first # to initialise the output.
  (2) Run $\mathcal{M}'$ (enumerating words), followed by $\mathcal{M}$ (to check if the current word is accepted). If $\mathcal{M}$ accepts w, then print w followed by #.

Then $\mathcal{E}$ enumerates exactly the words of **L** in some order of non-decreasing length.

Enumerators

Now suppose **L** can be enumerated by some TM $\mathcal{E}$ in some order of non-decreasing length.

- If **L** is finite, then **L** is accepted by a finite automaton.
- If **L** is infinite, then we define a decider $\mathcal{M}$ for it as follows.

$\mathcal{M}$:= On input w

- Simulate $\mathcal{E}$ until it either outputs w or some word longer than w
- If $\mathcal{E}$ outputs w, then accept, else reject.

Observation: since **L** is infinite, for each $w \in \Sigma^*$ the TM $\mathcal{E}$ will eventually generate w or some word longer than w. Therefore, $\mathcal{M}$ always halts and thus decides **L**.

□

Summary and Outlook

Turing Machines are a simple model of computation

Recognisable (semi-decidable) = recursively enumerable

Decidable = computable = recursive

Many variants of TMs exist – they normally recognise/decide the same languages

What's next?

- A short look into undecidability
- Recursion and self-referentiality
- Actual complexity classes