Technische Universität Dresden

Prof. Dr. Sebastian Rudolph

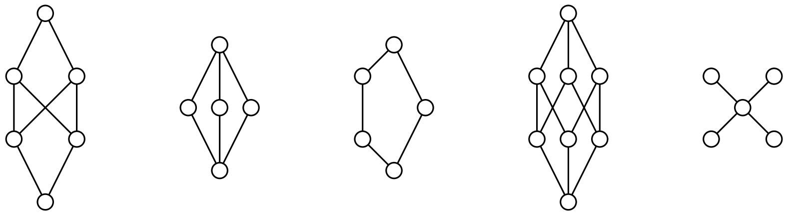
Introduction to Formal Concept Analysis
Exercise Sheet 2, Winter Semester 2017/18

Exercise 1 (line diagram)

a) Define: What is a lattice?

b) Find a preferably small lattice and draw its line diagram.

c) Which of the following line diagrams does not represent a lattice? Why?



(i) (ii) (iii) (iv) (v)

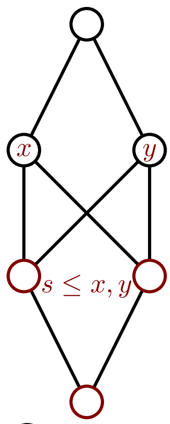
Solution:

a) A pair $(M, \leq)$ consisting of a set M together with a *reflexive*, *transitive*, and *antisymmetric* relation $\leq$ is called *ordered set*. For a subset A of M an element $s \in M$ is an *upper (lower) bound* of A, if $s \leq a$ $(a \leq s)$ holds for all $a \in A$. If a largest (smallest) upper (lower) bound of A exists, then it is called *infimum* (*supremum*) – in symbols: $\inf A$ $(\sup A)$ or $\bigwedge A$ $(\bigvee A)$. For two-elemental subsets $\{x, y\} \subseteq M$ we simply write $x \wedge y$ $(x \vee y)$.

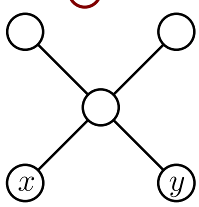
A *lattice* $\mathbb{V}$ is an ordered set $(V, \leq)$ such that $x \wedge y$ and $x \vee y$ exist for all $x, y \in V$.

b) Choose, for example $\mathbb{V} := (\{0\}, \{(0, 0)\})$.

c) (i) does not represent a lattice, because, for instance, there exists no infimum for the elements $\{x, y\}$:



(v) does not represent a lattice, because, for instance, there exists no infimum for the elements $\{x, y\}$:



Exercise 2 (complete lattice)

a) Define: What is a complete lattice?

b) Can you find a *complete* lattice among the lattices of Exercise 1c?

c) Let $P := (M, \leq)$ be an ordered set such that for every subset X of M the infimum $\bigwedge X$ exists. Show that P is a complete lattice.

Solution:

a) An ordered set $\mathbb{V} := (V, \leq)$ is a *complete lattice* if $\bigvee X$ and $\bigwedge X$ exist for all subsets X of V.

b) Every *finite* lattice is also a complete lattice.

c) We have to show that $\bigvee X$ exists for all subsets X of M.

For $X \subseteq M$ we set $S := \{s \in M \mid \forall a \in X \colon a \leq s\}$ (S is the set of upper bounds of X). It holds that $S \subseteq M$ and hence also $\underbrace{\bigwedge S}_{=:\perp} \in M$.

Furthermore, every element $a \in X$ is a lower bound of S and hence $a \leq \perp$ holds for all $a \in X$. Therefore, $\perp \in S$ holds and thus also $\perp = \bigvee X$.

Exercise 3

Prove the following theorem:

Let $(L, \leqslant)$ be a lattice with supremum and infimum defined as usual. For any elements $x, y, z \in L$ holds:

(i) $x \wedge y = y \wedge x$	(ii) $x \vee y = y \vee x$
(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$	(iv) $x \vee (y \vee z) = (x \vee y) \vee z$
(v) $x \wedge (x \vee y) = x$	(vi) $x \vee (x \wedge y) = x$
(vii) $x \wedge x = x$	(viii) $x \vee x = x$

Solution:

First note that the proofs for equations (ii), (iv), (vi), and (viii) can be directly obtained from the proofs of the equations (i), (iii), (v), and (vii), respectively, by duality. We now prove those.

(i) $x \wedge y = y \wedge x$
$x \wedge y = \bigwedge\{x, y\} = \bigwedge\{y, x\} = y \wedge x$

(iii) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
Obviously, for any two lattice elements u and v holds that they are equal if and only if their sets of lower bounds are the same. Thus we can prove the above equality by showing that any element w is a lower bound of $x \wedge (y \wedge z)$ exactly if it is a lower bound of $(x \wedge y) \wedge z$:

$w \leq x \wedge (y \wedge z) \Leftrightarrow w \leq x,\ w \leq (y \wedge z) \Leftrightarrow w \leq x,\ w \leq y,\ w \leq z \Leftrightarrow w \leq x \wedge y,\ w \leq z \Leftrightarrow w \leq (x \wedge y) \wedge z$

(v) $x \wedge (x \vee y) = x$
We first show that $u \leq v$ implies $u \wedge v = u$:
If $u \leq v$, then the lower bounds of u are a subset of the lower bounds of v. Consequently the common lower bounds of u and v are just the lower bounds of u therefore, the greatest common lower bound of u and v is the greatest lower bound of u which is u.

Now we are ready to prove the above equality: First we see that $x \vee y \geq x$, since every common upper bound of x and y is an upper bound of x and hence must be greater than x. Then, using the correspondence shown before, we obtain $x \wedge (x \vee y) = x$

(vii) $x \wedge x = x$
$x \wedge x = \bigwedge\{x, x\} = \bigwedge\{x\} = x$

Exercise 4 (the basic theorem of formal concept analysis)

1. Show that $\underline{L} := (L, |)$ with $L := \{2^i 3^j 5^k \in \mathbb{N} \mid 0 \le i,j \le 2; 0 \le k \le 3\}$ is a complete lattice ($a|b$ is shorthand for the relation "a divides b").
2. Draw the line diagram for $\underline{L}$.
3. Which are the supremum-irreducible elements?
4. Which are the infimum-irreducible elements?
5. Give a formal context (G, M, I) such that its concept lattice is isomorphic to $\underline{L}$. Give the isomorphism explicitly.
6. How could the fact that $\underline{L}$ and $\mathfrak{B}(G, M, I)$ are isomorphic be shown using the Basic Theorem of Formal Concept Analysis?

<u>Solution:</u>

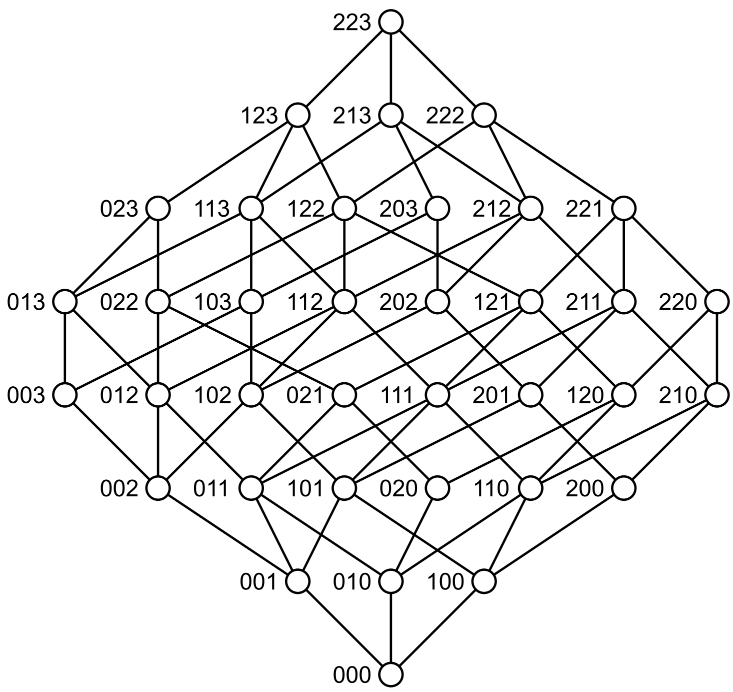
1) Since $\underline{L}$ is finite it is sufficient to show that $x \vee y$ and $x \wedge y$ exist for all $x, y \in \underline{L}$. Let us first state that

$$2^i 3^j 5^k | 2^\ell 3^m 5^n \text{ if and only if } i \le \ell \text{ and } j \le m \text{ and } k \le n$$

holds. From that it follows that:

$$\begin{aligned} 2^i 3^j 5^k \wedge 2^\ell 3^m 5^n &= 2^{\min(i,\ell)} 3^{\min(j,m)} 5^{\min(k,n)} \\ 2^i 3^j 5^k \vee 2^\ell 3^m 5^n &= 2^{\max(i,\ell)} 3^{\max(j,m)} 5^{\max(k,n)} \end{aligned}$$

2) We are writing ijk as shorthand for $2^i 3^j 5^k$.



3, 4) An element of a finite lattice is $\vee$-irreducible ($\wedge$-irreducible), if and only if the corresponding node in the line diagram has exactly one lower (upper) neighbor. Thus, the following propositions hold:

- $\wedge$-irreducible elements are exactly the numbers $2^i3^j5^k$ with exactly two *maximal* exponents.
- $\vee$-irreducible elemente are exactly the numbers $2^i3^j5^k$ with exactly two *minimal* exponents. These are exactly the prime powers.

5) We set $\mathbb{K} := (L, L, |)$. We have to show that $\underline{\mathfrak{B}}(L, L, |)$ and $\underline{L}$ are isomorphic, i.e., there exists a bijective map $\varphi\colon \underline{\mathfrak{B}}(L, L, |) \to \underline{L}$ with

$$(A_1, B_1) \leq (A_2, B_2) \text{ if and only if } \varphi(A_1, B_1) | \varphi(A_2, B_2).$$

We now show that $\varphi\colon (A, B) \mapsto A \cap B$ is an isomorphism between $\underline{\mathfrak{B}}(L, L, |)$ and $\underline{L}$. Therefore, we use the following abbreviations:

$$\begin{aligned} \{ijk\}\!\downarrow &:= \{2^\ell 3^m 5^k \mid \ell \leq i, m \leq j, n \leq k\} \\ \{ijk\}\!\uparrow &:= \{2^\ell 3^m 5^k \mid \ell \geq i, m \geq j, n \geq k\} \end{aligned}$$

Now $(A, B) \in \underline{\mathfrak{B}}(L, L, I)$ holds if and only if exponents $0 \leq i \leq 2, 0 \leq j \leq 2, 0 \leq k \leq 3$ exist such that $A = \{ijk\}\!\downarrow$ and $B = \{ijk\}\!\uparrow$. This allows us to show that φ is surjective and injective and thus bijective.

Furthermore, for two arbitrary concepts $(A_1, B_1) =: (\{ijk\}\!\downarrow, \{ijk\}\!\uparrow)$ and $(A_2, B_2) =: (\{\ell mn\}\!\downarrow, \{\ell mn\}\!\uparrow)$ the following holds:

$$\begin{aligned} (A_1, B_1) \leq (A_2, B_2) :\iff\; & (\{ijk\}\!\downarrow, \{ijk\}\!\uparrow) \leq (\{\ell mn\}\!\downarrow, \{\ell mn\}\!\uparrow) \\ \iff\; & \{ijk\}\!\downarrow \subseteq \{\ell mn\}\!\downarrow \\ \iff\; & i \leq \ell \wedge j \leq m \wedge k \leq n \\ \iff\; & \underbrace{2^i 3^j 5^k}_{=\varphi(A_1, B_1)} \mid \underbrace{2^\ell 3^m 5^n}_{=\varphi(A_2, B_2)} \end{aligned}$$

6) According to the Basic Theorem of Formal Concept Analysis, we need to specify two maps $\tilde{\gamma}\colon L \to L$ and $\tilde{\mu}\colon L \to L$, such that $\tilde{\gamma}(L)$ is supremum-dense and $\tilde{\mu}(L)$ infimum-dense in $\underline{L}$ and $a|b$ holds if and only if $\tilde{\gamma}(a)|\tilde{\mu}(b)$ holds. This is the case with the identity function, respectively, i.e., $\tilde{\mu}\colon l \mapsto l$ and $\tilde{\gamma}\colon l \mapsto l$ for $l \in L$.