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Complexity Theory

Exercise 2: Undecidability and Rice's Theorem

Stephan's Birthday (29th October 2024)

Exercise 2.1. Using an oracle that decides the halting problem, construct a decider for the language $\{ \langle \mathcal{M}, w \rangle \mid \mathcal{M} \text{ is a TM that accepts } w \}$.

Exercise 2.2. A *useless state* in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing machine has any useless states. Show that this language is undecidable.

Exercise 2.3. Show the following: "If a language $\mathbf{L}$ is Turing-recognisable and $\overline{\mathbf{L}}$ is many-one reducible to $\mathbf{L}$, then $\mathbf{L}$ is decidable."

Exercise 2.4. For this task assume an alphabet Σ with $|\Sigma| > 1$. Let

$$\mathbf{L} = \{ \langle \mathcal{M} \rangle \mid \mathcal{M} \text{ a TM that accepts } w^r \text{ whenever it accepts } w \text{ (for all } w \in \Sigma^*) \},$$

where w^r is the word w reversed. Show that $\mathbf{L}$ is undecidable.

Exercise 2.5. Consider the following languages $\mathbf{L}$ and $\mathbf{L}'$:

$$\mathbf{L} = \{ \langle \mathcal{M}, w \rangle \mid \mathcal{M} \text{ is a TM that accepts } w \}$$
$$\mathbf{L}' = \{ \langle \mathcal{M} \rangle \mid \mathcal{M} \text{ is a TM that does not accept any word} \}$$

Show that there cannot exist a many-one reduction from $\mathbf{L}$ to $\mathbf{L}'$.

Exercise 2.6. Show that every Turing-recognisable language can be mapping-reduced to the following language.

$$\{\langle \mathcal{M}, w \rangle \mid \mathcal{M} \text{ is a TM that accepts the word } w\}$$