TECHNISCHE UNIVERSITÄT DRESDEN

International Center for Computational Logic

COMPLEXITY THEORY

Lecture 1: Introduction and Motivation

Markus Krötzsch

Knowledge-Based Systems

TU Dresden, 14th Oct 2024

More recent versions of this slide deck might be available.
For the most current version of this course, see
`https://iccl.inf.tu-dresden.de/web/Complexity_Theory/en`

Course Tutors

Markus Krötzsch
Prof

Stephan Mennicke
Exercises

Organisation

Lectures

Monday, DS 2 (9:20–10:50), APB E009
Tuesday, DS 2 (9:20–10:50), APB E005

Exercise Sessions (starting 22 October)

Tuesday, DS 5 (14:50–16:20), APB E005

Web Page

`https://iccl.inf.tu-dresden.de/web/Complexity_Theory_(WS2024)`

Lecture Notes

Slides of current and past lectures will be online.

Goals and Prerequisites

Goals

- Introduce basic notions of **computational complexity theory**
- Introduce **commonly known complexity classes** (P, NP, PSpace, . . .) and discuss relationships between them
- Develop **tools to classify problems** into their corresponding complexity classes
- Introduce some **advanced topics of complexity theory** (e.g., circuits, probabilistic computation, quantum computing)

(Non-)Prerequisites

- No particular prior courses needed
- Prior acquaintance with Turing Machines and basic topics in formal languages and complexity is helpful
- General mathematical and theoretical computer science skills necessary

Reading List

- **Michael Sipser: Introduction to the Theory of Computation, International Edition; 3rd Edition; Cengage Learning 2013**
- Sanjeev Arora and Boaz Barak: **Computational Complexity: A Modern Approach**; Cambridge University Press 2009
- Michael R. Garey and David S. Johnson: **Computers and Intractability**; Bell Telephone Laboratories, Inc. 1979
- Erich Grädel: **Complexity Theory**; Lecture Notes, Winter Term 2009/10
- John E. Hopcroft and Jeffrey D. Ullman: **Introduction to Automata Theory, Languages, and Computation**; Addison Wesley Publishing Company 1979
- Christos H. Papadimitriou: **Computational Complexity**; 1995 Addison-Wesley Publishing Company, Inc

Computational Problems are Everywhere

Example 1.1:

- What are the factors of 54,623?
- What is the shortest route by car from Berlin to Hamburg?
- My program now runs for two weeks. Will it ever stop?
- Is this C++ program syntactically correct?

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Approach to problems:

> [T]he way is to avoid what is strong, and strike at what is weak.

(Sun Tzu: The Art of War, Chapter 6: Weak Points and Strong)

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Observation

Difficulty of a problem is hard to assess

Measuring the Difficulty of Problems

Question

How can we measure the complexity of a problem?

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Approach

Estimate the resource requirements of the “best” algorithm that solves this problem.

Typical Resources:

- Running Time
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Note

To assess the complexity of a problem, we need to consider **all possible algorithms** that solve this problem.

Problems

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Example 1.4: "Problem: Is a given graph connected?" will be modelled as the word problem of the language

$$\text{GCONN} := \{ \langle G \rangle \mid G \text{ is a connected graph} \}.$$

Then for a graph G we have

$$G \text{ is connected} \iff \langle G \rangle \in \text{GCONN}.$$

Note

The notation $\langle G \rangle$ denotes a suitable encoding of the graph G over some fixed alphabet (e.g., $\{ 0, 1 \}$).

Algorithms

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- Lambda Calculus
- μ-Recursion
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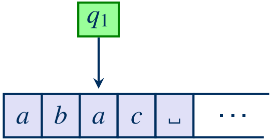

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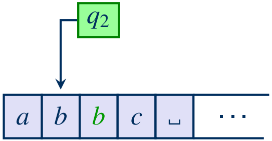

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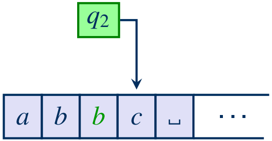

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Avoid: We will focus mostly on decidable problems in this course.

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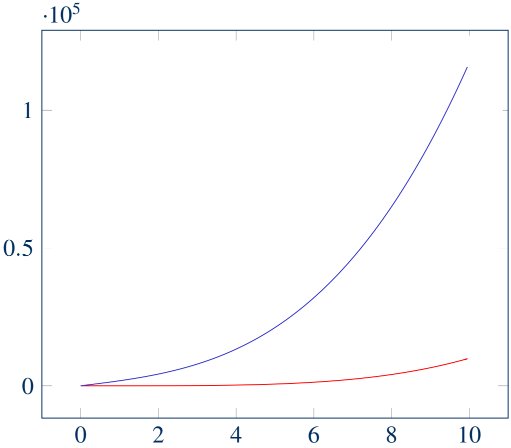
More formally:

$$f(n) = \mathcal{O}(g(n)) \iff \exists c > 0\, \exists n_0 \in \mathbb{N}\, \forall n > n_0 : f(n) \leq c \cdot g(n).$$

Big-$\mathcal{O}$-Notation: Example

$100n^3 + 1729n = \mathcal{O}(n^4)$:



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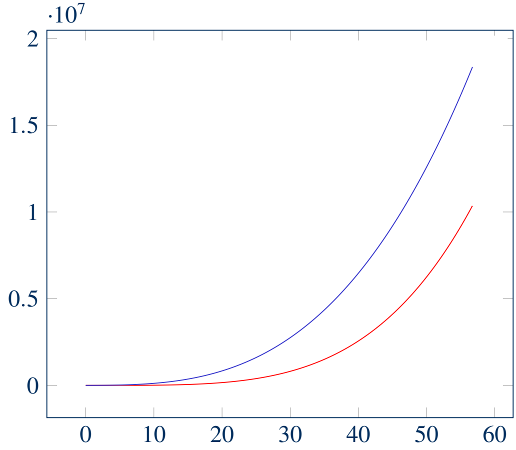

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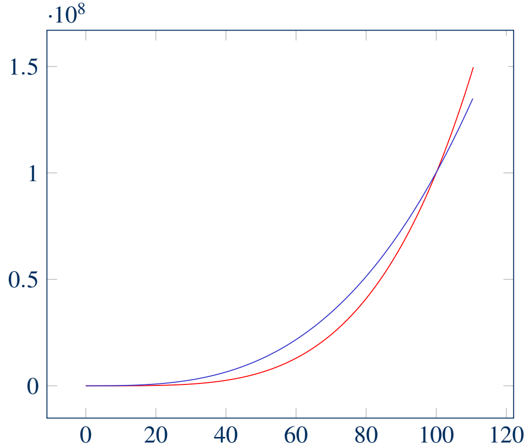

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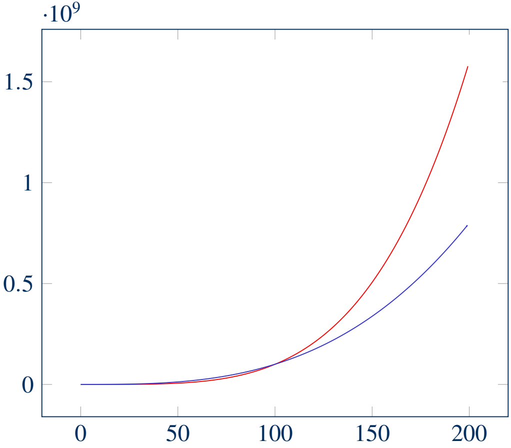

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The **time (space) complexity** of a problem is the asymptotic running time of a fastest (least memory consumptive) algorithm that solves the problem.

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- Exact complexity of TSP unknown

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$\oplus$P, #P, AC, AC^0, ACC0, AM, AP, APSpace, BPL, BPP, BQP, coNP, E, FP, IP, MA, MIP, NC, NExpTime, P$/$poly, PH, PP, RL, RP, Σ_i^{p}, TISP$(T(n), S(n))$, ZPP, ...

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Note

The Cobham-Edmonds-Thesis is only a **rule of thumb**: there are (practically) tractable problems outside of P, and (practically) intractable problems in P.

Friend or Foe?

Caveat

It is not known how big P is.

In particular, it is unknown whether $\mathsf{P} \neq \mathsf{NP}$ or not.

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But: we still do not know whether we can or cannot solve satisfiability efficiently. We only know it will be difficult to find out . . .

Learning Goals

- Get an overview over the foundations of Complexity Theory
- Gain insights into advanced techniques and results in Complexity Theory
- Understand what it means to "compute" something, and what the strengths and limits of different computing approaches are
- Get a feeling of how hard certain problems are, and where this hardness comes from
- Appreciate how very little we actually know about the computational complexity of many problems

Lecture Outline (1)

- **Turing Machines** (Revision)
  Definition of Turing Machines; Variants; Computational Equivalence; Decidability and Recognizability; Enumeration; Oracles
- **Undecidability**
  Examples of Undecidable Problems; Mapping Reductions; Rice's Theorem; Recursion Theorem
- **Time Complexity**
  Measuring Time Complexity; Many-One Reductions; Cook-Levin Theorem; Time Complexity Classes (P, NP, ExpTime); NP-completeness; pseudo-NP-complete problems
- **Space Complexity**
  Space Complexity Classes (PSpace, L, NL); Savitch's Theorem; PSpace-completeness; NL-completeness; NL = coNL

Lecture Outline (2)

- **Diagonalisation**
  Hierarchy Theorems (det. Time, non-det. Time, Space); Gap Theorem; Ladner's Theorem; Relativisation; Baker-Gill-Solovay Theorem
- **Alternation**
  Alternating Turing Machines; APTime = PSpace; APSpace = ExpTime; Polynomial Hierarchy
- **Circuit Complexity**
  Boolean circuits; alternative proof of Cook-Levin Theorem; parallel computation (NC); P-completeness; P/poly; (Karp-Lipton Theorem, Meyer's Theorem)
- **Probabilistic Computation**
  Randomised complexity classes (RP, PP, BPP, ZPP); Sipser-Gács-Lautemann Theorem
- **Quantum Computing**
  Quantum mechanics for computer scientists, entanglement, quantum circuits, BQP
- **Interactive Proofs**
  Prover and verifier; deterministic proof systems; probabilistic verifiers; the class IP

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